ADJOINT SELMER GROUPS OF AUTOMORPHIC GALOIS REPRESENTATIONS OF UNITARY TYPE

JAMES NEWTON AND JACK A. THORNE

Abstract. Let ρ be the p-adic Galois representation attached to a cuspidal, regular algebraic automorphic representation of GL_n of unitary type. Under very mild hypotheses on ρ, we prove the vanishing of the (Bloch–Kato) adjoint Selmer group of ρ. We obtain definitive results for the adjoint Selmer groups associated to non-CM Hilbert modular forms and elliptic curves over totally real fields.

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Introduction

Context. Let F be a CM number field, with maximal totally real subfield F^+. Fix an algebraic closure $\overline{F}$ of F and a complex conjugation $c \in \mathrm{Gal}(\overline{F}/F^+)$. We say that a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ is of unitary type if it is conjugate self-dual, i.e. if it satisfies the relation $\pi^c \cong \pi^\vee$. If π is conjugate self-dual and moreover regular algebraic (a condition on π_∞), then for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ there is an associated p-adic Galois representation

$$r_{\pi,\iota} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p),$$

characterized up to isomorphism by compatibility with the local Langlands correspondence at each finite place of F. The conjugate self-duality of π implies the existence of an isomorphism $r_{\pi,\iota}^c \cong r_{\pi,\iota}^\vee \otimes \epsilon^{1-n}$, where ϵ is the p-adic cyclotomic character.

This paper concerns the adjoint Bloch–Kato Selmer group of such a representation. To define it, we note that if V denotes the space on which $r_{\pi,\iota}$ acts, then the conjugate self-duality of $r_{\pi,\iota}$ is reflected in the existence of a perfect, symmetric, and Galois equivariant bilinear pairing

$$\langle \cdot, \cdot \rangle : V^c \times V \otimes \epsilon^{n-1} \to \overline{\mathbf{Q}}_p.$$

The existence of this pairing allows us to extend the adjoint action of $r_{\pi,\iota}$ on $\mathrm{End}(V)$ to an action of $\mathrm{Gal}(\overline{F}/F^+)$, where $c \in \mathrm{Gal}(\overline{F}/F^+)$ acts by the formula $c \cdot X = -X^*$ (and X^* is the adjoint with respect to $\langle \cdot, \cdot \rangle$).

We are interested in is the Bloch–Kato Selmer group

$$H^1_f(F^+, \mathrm{End}(V)) = \left\{ x \in H^1(F^+, \mathrm{End}(V)) : x_v \in H^1_f(F^+_v, \mathrm{End}(V)) \text{ for all finite places } v \right\}$$

where $H^1_f(F^+_v, \mathrm{End}(V))$ is $\ker\left(H^1(F^+_v, \mathrm{End}(V)) \to H^1(F^+_v, \mathrm{End}(V) \otimes_{\mathbf{Q}_p} B_{crys})\right)$ for $v|p$ and $H^1_{ur}(F^+_v, \mathrm{End}(V))$ for $v \nmid p$.

General conjectures predict the vanishing of this group (see the introduction of [All16] for a detailed discussion of this in the present context). We are content here to note that this group parameterizes infinitesimal deformations of $r_{\pi,\iota}$ which are at the same time conjugate self-dual and geometric, in the sense of p-adic Hodge theory.

Our results. The following is the main theorem of this paper.

Theorem A. *Let F be a CM number field, and let π be a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ of unitary type. Let p be a prime, and let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Suppose that $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous, in the sense of Definition 2.27. Then $H^1_f(F^+, \mathrm{ad}\, r_{\pi,\iota}) = 0$.*

For some examples of enormous subgroups, see §2.32. For example, we note that our condition is satisfied for any π such that for some finite place v of F, π_v is a twist of the Steinberg representation.

We compare Theorem A with some other results in the literature that are proved using related techniques. Kisin [Kis04] proved the analogue of Theorem A for the Galois representations attached to classical holomorphic modular forms under some mild conditions on the residual representation. Allen [All16] proved a result similar to Theorem A, but assuming a stronger condition on the residual representation $\overline{r}_{\pi,\iota}$, requiring in particular that it be irreducible (similar results were also obtained by Breuil–Hellmann–Schraen [BHS17]). These works use variants of the Taylor–Wiles method, which is a powerful tool for studying the deformation theory of automorphic Galois representations.

Our main motivation for this work was to prove a result valid under very weak conditions on the residual representation. In particular, we allow the case $p = 2$ and $\overline{r}_{\pi,\iota}$ trivial, which is rather far from the cases allowed by [All16]. For example, we obtain the following results for 2-dimensional representations over totally real fields.

Theorem B. *Let F be a totally real number field, and let p be a prime.*

(1) Let π be a non-CM, regular algebraic automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Then for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, $H^1_f(F, \mathrm{ad}\, r_{\pi,\iota}) = 0$.

(2) Let E be a non-CM elliptic curve over F, and let $r_p(E) : G_F \to \mathrm{GL}_2(\mathbf{Q}_p)$ denote the associated p-adic representation. Then $H^1_f(F, \mathrm{ad}\, r_p(E)) = 0$.

We emphasise that no additional conditions are required in either case in order to conclude the vanishing of the adjoint Selmer group.

There are three main innovations that allow us to prove a result like Theorem A. The first is a control theorem for studying the pseudodeformation theory of a representation $\rho : \Gamma \to \mathrm{GL}_n(\mathbf{Z}_p)$ of a profinite group Γ. We recall that ρ has an associated pseudocharacter $\mathrm{tr}\, \rho$, which can be defined following either Chenevier

[Che14] or Lafforgue [Laf18] (the proof that these two notions are equivalent being due to Emerson [Eme18]). If the residual representation $\overline{\rho}$ is absolutely irreducible then it is known that deforming $\operatorname{tr} \rho$ is equivalent to deforming ρ. In general any deformation of ρ gives rise to a deformation of $\operatorname{tr} \rho$, but the two notions are not equivalent.

Here we use Lafforgue's definition of pseudocharacter to show that that if $\rho \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is absolutely irreducible, then there is a reasonably strong link between deformations and pseudodeformations with coefficients in the ring $\mathbf{Z}_p \oplus \epsilon \mathbf{Z}_p/(p^N)$. Informally, deformations and pseudodeformations are "the same", up to bounded torsion which depends only on the image $\rho(\Gamma)$. See Proposition 2.9 for a precise statement.

The second innovation is the formulation, by Wake and Wang-Erickson [WWE19], of functors of pseudodeformations satisfying deformation conditions (e.g. conditions arising from p-adic Hodge theory). This is an indispensable tool for making an effective comparison between pseudodeformation rings and Hecke algebras acting on classical automorphic forms.

The third innovation is related to the use of Taylor–Wiles systems in our proof. To make use of Taylor–Wiles systems in the study of automorphic forms with integral coefficients, one needs to show that if q is a Taylor–Wiles place, then the space of automorphic forms with unramified level at q is isomorphic to the space of automorphic forms with Iwahori level at q, after localization with respect to a suitable eigenvalue of the U_q operator (see for example [CG18, Lemma 5.8]). One can argue along these lines only if the residual representation $\overline{\rho}$, unramified at q by hypothesis, has the property that $\overline{\rho}(\mathrm{Frob}_q)$ has distinct eigenvalues. This is the reason for condition (2) in the statement of [Kis04, Introduction, Theorem] and of course excludes the case where $\overline{\rho}$ is trivial. Without this "independence of q" statement, one does not have the finiteness conditions needed to carry out the Taylor–Wiles patching argument, at least as outlined in [Dia97].

In his thesis, Pan [Pan19] introduced a surprising technique to circumvent this issue. Building on Scholze's interpretation of the Taylor–Wiles patching argument using ultrafilters [Sch18], Pan constructs a huge "pre-patched module", and then shows that using suitable Hecke operators it can be cut down to a size making it suitable for use in the Taylor–Wiles argument. We have adapted his arguments to our context (in some ways more elementary, since we work with fixed weight automorphic forms, whereas [Pan19] works with completed cohomology).

Applications. Results such as Theorem A have applications to the geometry of eigenvarieties, and this is one of the main motivations for proving them (as was already the case for Kisin [Kis04]). This is because one can embed (at least locally around an irreducible point) eigenvarieties inside deformation spaces of trianguline representations. In many cases, the vanishing of $H^1_f(F^+, \operatorname{ad} r_{\pi,\iota})$ can be used to prove that this embedding is in fact a local isomorphism.

For example, the vanishing of the adjoint Selmer group is a significant part of what it means for a p-refined Hilbert modular form to be decent, in the sense of [BH17], and therefore to admit a p-adic L-function with good interpolation properties. Another application is that one can use an understanding of the geometry of the eigenvariety to prove modularity results for Galois representations. This possibility is already suggested in Kisin's work [Kis03, (11.13)]. We will take this point of view in [NT], where Theorem A is one of the key inputs to prove the automorphy of the symmetric power liftings of level one Hecke eigenforms (for example, Ramanujan's modular form Δ).

Organization of this paper. In Section 2 we establish our control theorem relating pseudodeformations and deformations (up to bounded torsion), and set up the Galois theoretic ingredients for the Taylor–Wiles method. In the short Section 3 we prove a simple representation-theoretic result which controls the difference between spaces of automorphic forms with hyperspecial and Iwahori level at Taylor–Wiles places. In Section 4 we carry out our variation on the Taylor–Wiles method (inspired by Pan's work) and prove a special case of Theorem A. Finally, the general case of Theorem A, together with Theorem B and some other applications are deduced in Section 5 using base change and potential automorphy.

Acknowledgements. J.T.'s work received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 714405). J.N. would like to thank Carl Wang-Erickson for helpful discussions about the work [WWE19]. We are grateful to the anonymous referee and to Florian Herzig for their detailed comments on this paper.

1. Notation and preliminaries

If F is a field of characteristic zero, we generally fix an algebraic closure $\overline{F}/F$ and write G_F for the absolute Galois group of F with respect to this choice. If F is a number field, then we will also fix embeddings $\overline{F} \to \overline{F}_v$ extending the map $F \to F_v$ for each place v of F; this choice determines a homomorphism $G_{F_v} \to G_F$. When v is a finite place, we will write $\mathcal{O}_{F_v} \subset F_v$ for the valuation ring, $\varpi_v \in \mathcal{O}_{F_v}$ for a fixed choice of uniformizer, $\mathrm{Frob}_v \in G_{F_v}$ for a fixed choice of Frobenius lift, $k(v) = \mathcal{O}_{F_v}/(\varpi_v)$ for the residue field, and $q_v = \#k(v)$ for the cardinality of the residue field. When v is a real place, we write $c_v \in G_{F_v}$ for complex conjugation. If S is a finite set of finite places of F then we write F_S/F for the maximal subextension of $\overline{F}$ unramified outside S and $G_{F,S} = \mathrm{Gal}(F_S/F)$.

If p is a prime, then we call a coefficient field a finite extension $E/\mathbf{Q}_p$ contained inside our fixed algebraic closure $\overline{\mathbf{Q}}_p$, and write $\mathcal{O}$ for the valuation ring of E, $\varpi \in \mathcal{O}$ for a fixed choice of uniformizer, and $k = \mathcal{O}/(\varpi)$ for the residue field. We write $\mathcal{C}_{\mathcal{O}}$ for the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k.

If A is a ring and $\rho : \Gamma \to \mathrm{GL}_n(A)$ is a representation, we write $\mathrm{ad}\,\rho$ for $M_n(A)$ with its adjoint Γ-action, and $\mathrm{ad}^0\,\rho \subset \mathrm{ad}\,\rho$ for the $A[\Gamma]$-submodule of trace 0 matrices. We will use the self-duality $\mathrm{ad}\,\rho \times \mathrm{ad}\,\rho \to A$, $(X, Y) \mapsto \mathrm{tr}\,XY$, to identify $\mathrm{ad}\,\rho$ with its dual when we e.g. define dual Selmer conditions using Tate duality (see for example §2.19).

If G is a locally profinite group and $U \subset G$ is an open compact subgroup, then we write $\mathcal{H}(G, U)$ for the set of compactly supported, U-biinvariant functions $f : G \to \mathbf{Z}$. It is a $\mathbf{Z}$-algebra, where convolution is defined using the left-invariant Haar measure normalized to give U measure 1; see [NT16, §2.2]. It is free as a $\mathbf{Z}$-module, with basis given by the characteristic functions $[UgU]$ of double cosets.

Let K be a non-archimedean characteristic 0 local field, and let Ω be an algebraically closed field of characteristic 0. We write $W_K \subset G_K$ for the Weil group of K and $I_K \subset W_K$ for the inertia subgroup. We use the cohomological normalisation of class field theory: it is the isomorphism $\mathrm{Art}_K : K^\times \to W_K^{ab}$ which sends uniformizers to geometric Frobenius elements. We use the Tate normalisation of the local Langlands correspondence for GL_n: it is the bijection rec_K^T between isomorphism classes of irreducible, admissible $\Omega[\mathrm{GL}_n(K)]$-modules and isomorphism classes Frobenius-semisimple Weil–Deligne representations over Ω of rank n which is normalised as in [CT14, §2.1].

If $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is a continuous representation (assumed to be de Rham if p equals the residue characteristic of K), then we write $\mathrm{WD}(\rho) = (r, N)$ for the associated Weil–Deligne representation, and $\mathrm{WD}(\rho)^{F-ss}$ for its Frobenius semisimplification.

Definition 1.1. *We say that a Weil–Deligne representation (r, N) is generic if there is no non-zero morphism $(r, N) \to (r(1), N)$. We say that a continuous representation ρ is generic if $WD(\rho)$ is generic.*

We note that if $WD(\rho)^{F-ss}$ is generic, then ρ is generic. It follows from [All16, Lemma 1.1.3] that if π is a generic irreducible admissible $\overline{\mathbf{Q}}_p[\mathrm{GL}_n(K)]$-module and $WD(\rho)^{F-ss} = \mathrm{rec}_K^T(\pi)$, then ρ is generic.

If p equals the residue characteristic of K and V is the E-vector space on which ρ acts (for some $E \subset \overline{\mathbf{Q}}_p$ finite over $\mathbf{Q}_p$ with $\rho(G_K) \subset \mathrm{GL}_n(E)$), we have subspaces

$$H^1_f(K, V) \subset H^1_g(K, V) \subset H^1(K, V)$$

defined by

$$\begin{aligned} H^1_f(K, V) &= \ker\left(H^1(K, V) \to H^1(K, V \otimes_{\mathbf{Q}_p} B_{crys})\right) \\ H^1_g(K, V) &= \ker\left(H^1(K, V) \to H^1(K, V \otimes_{\mathbf{Q}_p} B_{dR})\right). \end{aligned}$$

We have $H^1_f(K, \mathrm{End}(V)) = H^1_g(K, \mathrm{End}(V))$ if and only if ρ is generic [All16, Remark 1.2.9]. Similarly, if p does not equal the residue characteristic of K, we have a subspace $H^1_{ur}(K, V) = \ker\left(H^1(K, V) \to H^1(I_K, V)\right)$. For notational compatibility with the p-adic case we write $H^1_f(K, V) = H^1_{ur}(K, V)$ and $H^1_g(K, V) = H^1(K, V)$. Then we again have $H^1_f(K, \mathrm{End}(V)) = H^1_g(K, \mathrm{End}(V))$ if and only if ρ is generic.

Let F be a number field, and let S be a finite set of finite places of F, containing the p-adic places S_p. Let $r : G_{F,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a continuous representation, with underlying E-vector space V. We have global Selmer groups

$$H^1_f(F, V) \subset H^1_{g,S}(F, V) \subset H^1(F_S/F, V)$$

defined by

$$\begin{aligned} H^1_f(F, V) &= \ker\left(H^1(F_S/F, V) \to \prod_{v \in S} H^1(F_v, V)/H^1_f(F_v, V)\right) \\ H^1_{g,S}(F, V) &= \ker\left(H^1(F_S/F, V) \to \prod_{v \in S} H^1(F_v, V)/H^1_g(F_v, V)\right) \\ &= \ker\left(H^1(F_S/F, V) \to \prod_{v \in S_p} H^1(F_v, V)/H^1_g(F_v, V)\right). \end{aligned}$$

We note our convention that $H^1(F_S/F, *)$ denotes group cohomology for the group $G_{F,S}$. The group $H^1_f(F, V)$ does not change when S is enlarged (this is why we do not record S in the notation).

If F is a number field and π is an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, we say that π is regular algebraic if π_∞ has the same infinitesimal character as an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$. We recall (cf. [BLGGT14, §2.1]) that if F is a totally real or CM number field, then a pair (π, χ) comprising an automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and a Hecke character $\chi : (F^+)^\times \backslash (\mathbf{A}_{F^+})^\times \to \mathbf{C}^\times$ is said to be polarized if there is an isomorphism $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+})$ and, if F is CM, then $\chi_v(-1) = (-1)^n$ for each place $v|\infty$ of F. (The sign condition of [BLGGT14] in the case that F is totally real can be suppressed, as a consequence of [Pat15, Theorem 2.1].) The automorphic representations of unitary type discussed in our introduction correspond to polarized automorphic representations $(\pi, \delta^n_{F/F^+})$, where δ_{F/F^+} is the quadratic character for F/F^+.

If (π, χ) is a regular algebraic, cuspidal, polarized automorphic representation, then for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ there is an associated Galois representation (we refer to [BLGGT14, Theorem 2.1.1] for its properties)

$$r_{\pi,\iota} : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p).$$

If F is CM, then $r_{\pi,\iota}$ extends to a homomorphism $r_{\pi,\iota} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbf{Q}}_p)$, with multiplier character $\nu \circ r_{\pi,\iota} = \epsilon^{1-n} r_{\chi,\iota}$ ($\mathcal{G}_n$ is the algebraic group defined in [CHT08, §2.1]; here the word 'extends' is interpreted following the convention described at the top of [CHT08, p. 8]). This defines an extension of the G_F action on $\mathrm{ad}\, r_{\pi,\iota}$ to an action of G_{F^+}. More explicitly, if we fix a choice $c \in G_{F^+}$ of complex conjugation, there is a perfect, symmetric pairing $\langle \cdot, \cdot \rangle$ on $\overline{\mathbf{Q}}_p^n$ such that

$$\langle r_{\pi,\iota}(\sigma) v, r_{\pi,\iota}(\sigma^c) w \rangle = (\epsilon^{1-n} r_{\chi,\iota}(\sigma)) \langle v, w \rangle$$

for all $\sigma \in G_F, v, w \in \overline{\mathbf{Q}}_p^n$ and c acts on $\mathrm{ad}\, r_{\pi,\iota} = \mathrm{End}(\overline{\mathbf{Q}}_p^n)$ by $X \mapsto -X^*$, where X^* is the adjoint with respect to $\langle \cdot, \cdot \rangle$.

2. Pseudocharacters

In this paper we use Lafforgue's notion of pseudocharacter for a reductive group in the case of GL_n (see [Laf18, §11] or [BHKT, §4]), and Chenevier's notion of group determinant [Che14]. In fact, these are equivalent, but both definitions are useful. We will prove a new result about the deformation theory of pseudocharacters (Proposition 2.9) using Lafforgue's point of view, while we follow [WWE19] in using Chenevier's definition to impose deformation conditions on pseudocharacters.

2.1. Pseudocharacters vs. determinants.

We begin by recalling the relevant definitions. Let Γ be a group and fix $n \geq 1$.

Definition 2.2. *A pseudocharacter of Γ of dimension n over a ring A is a collection $\Theta = (\Theta_m)_{m \geq 1}$ of algebra homomorphisms $\Theta_m : \mathbf{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n} \to \mathrm{Map}(\Gamma^m, A)$ satisfying the following conditions:*

(1) *For all $k, l \geq 1$ and for each map $\zeta : \{1, \ldots, k\} \to \{1, \ldots, l\}$, each $f \in \mathbf{Z}[\mathrm{GL}_n^k]^{\mathrm{GL}_n}$, and each $\gamma_1, \ldots, \gamma_l \in \Gamma$, we have*

$$\Theta_l(f^\zeta)(\gamma_1, \ldots, \gamma_l) = \Theta_k(f)(\gamma_{\zeta(1)}, \ldots, \gamma_{\zeta(k)}),$$

where $f^\zeta(g_1, \ldots, g_l) = f(g_{\zeta(1)}, \ldots, g_{\zeta(k)})$.

(2) *For each $k \geq 1$, for each $\gamma_1, \ldots, \gamma_{k+1} \in \Gamma$, and for each $f \in \mathbf{Z}[\mathrm{GL}_n^k]^{\mathrm{GL}_n}$, we have*

$$\Theta_{k+1}(\hat{f})(\gamma_1, \ldots, \gamma_{k+1}) = \Theta_k(f)(\gamma_1, \ldots, \gamma_k \gamma_{k+1}),$$

where $\hat{f}(g_1, \ldots, g_{k+1}) = f(g_1, \ldots, g_k g_{k+1})$.

If $\rho : \Gamma \to \mathrm{GL}_n(A)$ is a representation, then we can define its associated pseudocharacter $t = (t_m)_{m \geq 1} = \operatorname{tr} \rho$ by the formula

$$t_m(f)(\gamma_1, \ldots, \gamma_n) = f(\rho(\gamma_1), \ldots, \rho(\gamma_m)).$$

One can define the operations of twisting and duality on pseudocharacters in a way compatible with the usual operations on representations. For example, let $i : \mathrm{GL}_n \to \mathrm{GL}_n$ be the involution given by $i(g) = {}^t g^{-1}$. If t is a pseudocharacter, then we define a new pseudocharacter $t^\vee$ by the formula $t_m^\vee(f)(\gamma_1, \ldots, \gamma_m) = t_m(f')(\gamma_1, \ldots, \gamma_m)$, where $f' \in \mathbf{Z}[\mathrm{GL}_n^m]$ is defined by

$$f'(g_1, \ldots, g_m) = f(i(g_1), \ldots, i(g_m)).$$

If $t = \operatorname{tr} \rho$, then $t^\vee = \operatorname{tr} \rho^\vee$.

Similarly, if $\chi : \Gamma \to A^\times$ is a character, then we define the twist $t \otimes \chi$ by the formula $(t \otimes \chi)_m(f)(\gamma_1, \ldots, \gamma_m) = f'(\gamma_1, \ldots, \gamma_m)$, where $f' \in A[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ is defined by the formula $f'(g_1, \ldots, g_m) = f(\chi_1(\gamma_1)g_1, \ldots, \chi_m(\gamma_m)g_m)$. If $t = \operatorname{tr} \rho$, then $t \otimes \chi = \operatorname{tr}(\rho \otimes \chi)$.

Before giving the definition of group determinant, we fix some notation. Let A be a ring and let A-alg be the category of commutative A-algebras. If M is an A-module, then we write $h_M : A\text{-alg} \to \mathrm{Sets}$ for the functor $B \mapsto M \otimes_A B$.

Definition 2.3. *A group determinant of Γ of dimension n over a ring A is a natural transformation of functors $D : h_{A[\Gamma]} \to h_A$ satisfying the following conditions on the induced map $B[\Gamma] \to B$ for every $B \in A$-alg:*

(1) $D(1) = 1$.
(2) For any $x, y \in B[\Gamma]$, $D(xy) = D(x)D(y)$.
(3) For any $x \in B[\Gamma]$, $b \in B$, $D(bx) = b^n D(x)$.

If $\rho : \Gamma \to \mathrm{GL}_n(A)$ is a representation, then we can define its associated group determinant $D(x) = \det(\rho(x))$ (where we extend ρ to a homomorphism $\rho : B[\Gamma] \to M_n(B)$ for any A-algebra B). We omit the formulae for the dual or twist of a group determinant.

We now describe the relation between pseudocharacters and group determinants. For each $i = 0, \ldots, n$, let $\lambda_i \in \mathbf{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n}$ be defined by the equation $\det(X - g) = \sum_{i=0}^n (-1)^i \lambda_i(g) X^{n-i}$. If t is a pseudocharacter, we have functions $(i = 0, \ldots, n)$

$$t^{[i]} : \Gamma \to A$$

given by the formulae $t^{[i]}(\gamma) = t_1(\lambda_i)(\gamma)$. By [Don92, §3.1], for any $m \geq 1$ $\mathbf{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ is generated as a ring by the functions $\lambda_i(g_{i_1} \ldots g_{i_r})$ $(r \in \mathbf{N}, 1 \leq i_1, \ldots, i_r \leq m)$, together with $\det(g_1 \ldots g_m)^{-1}$. The axioms defining a pseudocharacter show that we have

$$t_m(\lambda_i(g_{i_1} \ldots g_{i_r}))(\gamma_1, \ldots, \gamma_m) = t_1(\lambda_i(g))(\gamma_{i_1} \ldots \gamma_{i_r}). \tag{2.3.1}$$

It follows that the functions $t^{[i]}$ $(i = 0, \ldots, n)$ together determine t.

If D is a group determinant, then we define functions $(i = 0, \ldots, n)$

$$D^{[i]} : \Gamma \to A$$

by the formula $D(X - \gamma) = \sum_{i=0}^n (-1)^i D^{[i]}(\gamma) X^{n-i}$ (evaluation of D over the ring $A[X]$). The functions $D^{[i]}$ $(i = 0, \ldots, n)$ together determine D (by Amitsur's formula, cf. [Che14, Lemma 1.12]).

Theorem 2.4. *For any group Γ and ring A, the pseudocharacters t of dimension n are in canonical bijection with the group determinants D of dimension n. This bijection is characterized by the equality $t^{[i]} = D^{[i]}$ for each $i = 0, \dots, n$.*

Proof. See [Eme18, Theorems 4.0.1 and 5.0.1], which explicitly construct a bijection between the two classes of objects. Suppose that t, D are associated. Then for any $\gamma \in \Gamma$, $t_1(\gamma)$ determines a ring homomorphism $\mathbf{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n} \to A$, hence a ring homomorphism $t_1(\gamma)[X] : \mathbf{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n} \otimes_{\mathbf{Z}} \mathbf{Z}[X] \to A[X]$. We may think of $\det(X - g)$ as an element of $\mathbf{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n} \otimes_{\mathbf{Z}} \mathbf{Z}[X]$, and the proof of [Eme18, Theorem 4.0.1] shows that if $\gamma \in \Gamma$ then $D(X - \gamma) = t_1(\gamma)[X](\det(X - g))$. This equality is equivalent to equalities $D^{[i]}(\gamma) = t^{[i]}(\gamma)$ $(i = 0, \dots, n)$. $\square$

We now discuss continuity. Suppose therefore that Γ is a profinite group and A is a topological ring. The definitions are as follows.

Definition 2.5. *Let $t = (t_m)_{m \geq 1}$ be a pseudocharacter. We say that t is continuous if for each $m \geq 1$, t_m takes values in the set $\mathrm{Map}_{cts}(\Gamma^m, A)$ of continuous functions $\Gamma^m \to A$.*

Definition 2.6. *Let D be a group determinant. We say that D is continuous if each function $D^{[i]}$ $(i = 0, \dots, n)$ is continuous.*

It is clear from the definitions that if $\rho : \Gamma \to \mathrm{GL}_n(A)$ is a continuous representation, then $\operatorname{tr} \rho$ is continuous as a pseudocharacter.

Proposition 2.7. *Let $t = (t_m)_{m \geq 1}$ and D be associated under the bijection of Theorem 2.4. Then t is continuous if and only if D is.*

Proof. In light of Theorem 2.4, it is enough to show that if t is a pseudocharacter such that each function $t^{[i]}$ $(i = 0, \dots, n)$ is continuous, then t is continuous. This is again a consequence of (2.3.1) and [Don92, §3.1]. $\square$

2.8. **Pseudocharacters vs. representations.** Now let p be a prime, let $E/\mathbf{Q}_p$ be a finite extension, and let Γ be a profinite group. Let $\rho : \Gamma \to \mathrm{GL}_n(\mathcal{O})$ be a continuous homomorphism which is absolutely irreducible over E. Let $t = (t_m)_{m \geq 1} = \operatorname{tr} \rho$ denote the pseudocharacter associated to ρ. We consider liftings of ρ and of t to the ring $A = \mathcal{O} \oplus \epsilon E/\mathcal{O}$ (with $\epsilon^2 = 0$). Clearly if $\rho' : \Gamma \to \mathrm{GL}_n(A)$ is a lifting of ρ, in the sense that $\rho' \bmod (\epsilon) = \rho$, then $t' = \operatorname{tr} \rho'$ is a lifting of t. We want to show that in fact deforming ρ in this way is not too far from deforming t.

We write $\alpha_k : A \to A$ for the $\mathcal{O}$-algebra homomorphism which acts as multiplication by p^k on the ideal $(\epsilon) \subset A$. We will prove:

Proposition 2.9. *There exists an integer $k_0 \geq 0$, depending only on $\rho(\Gamma)$, with the following properties:*

1. *For any lifting t' of t to A, there exists a homomorphism $\rho' : \Gamma \to \mathrm{GL}_n(A)$ lifting ρ such that $\operatorname{tr} \rho' = \alpha_{k_0} \circ t'$. If t' is continuous, then ρ' can be chosen to be continuous.*
2. *If $\rho'_1, \rho'_2 : \Gamma \to \mathrm{GL}_n(A)$ are two liftings with $\operatorname{tr} \rho'_1 = \operatorname{tr} \rho'_2$, then $\alpha_{k_0} \circ \rho'_1$ and $\alpha_{k_0} \circ \rho'_2$ are conjugate under the action of the group $1 + \epsilon M_{n \times n}(E/\mathcal{O}) \subset \mathrm{GL}_n(A)$; and if $X \in M_{n \times n}(E/\mathcal{O})$ is such that $1 + \epsilon X$ centralizes ρ'_1, then $p^{k_0} X$ is a scalar matrix.*

For any $m \geq 1$, we define $X_m = (\mathrm{GL}_{n,\mathcal{O}})^m$, and $Y_m = \operatorname{Spec} \mathcal{O}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$. We write $\pi_m : X_m \to Y_m$ for the tautological morphism. We fix elements $\gamma_1, \dots, \gamma_m \in \Gamma$

such that $\rho(\gamma_1), \ldots, \rho(\gamma_m)$ generate a Zariski dense subgroup of $\rho(\Gamma)$. We may assume that $m \geq 2$.

Let
$$x = (g_1, \ldots, g_m) = (\rho(\gamma_1), \ldots, \rho(\gamma_m)) \in X_m(\mathcal{O}).$$
If $\gamma, \delta \in \Gamma$, then we define
$$x(\gamma) = (\rho(\gamma_1), \ldots, \rho(\gamma_m), \rho(\gamma)) \in X_{m+1}(\mathcal{O})$$
and
$$x(\gamma, \delta) = (\rho(\gamma_1), \ldots, \rho(\gamma_m), \rho(\gamma), \rho(\delta)) \in X_{m+2}(\mathcal{O}).$$
We define $y = \pi_m(x)$, $y(\gamma) = \pi_{m+1}(x(\gamma))$, and $y(\gamma, \delta) = \pi_{m+2}(x(\gamma, \delta))$. Before going further, we recall the following lemma.

Lemma 2.10. *Let $\pi : X \to Y$ be a separated morphism of schemes of finite type over a base S. Let G be a separated group scheme, smooth and of finite type over S, and suppose that G acts on X in such a way that π is G-equivariant for the trivial action of G on Y. Then:*

1. *There is a canonical morphism $\Omega_{X/Y} \to \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Lie}\, G, \mathcal{O}_X)$ of coherent sheaves of $\mathcal{O}_X$-modules.*
2. *If π is a G-torsor, then this morphism is an isomorphism.*

Proof. Let $e : X \to G \times X$ be the morphism $x \mapsto (e, x)$, and let $\mu : G \times X \to X \times_Y X$ be the morphism $(g, x) \mapsto (x, gx)$. Then $\mu \circ e$ is the diagonal embedding, and both e and $\mu \circ e$ are closed immersions. The sheaf $\mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Lie}\, G, \mathcal{O}_X)$ may be identified with the conormal sheaf of the morphism e (see e.g. [GP11, II, Lemme 4.11.7]) while $\Omega_{X/Y}$ may be identified with the conormal sheaf of the morphism $\mu \circ e$. The existence of the morphism therefore follows from [Sta13, Lemma 01R4].

Now suppose that π is a G-torsor. In this case μ is an isomorphism, and the statement is immediate. $\square$

Let $\mathfrak{g} = \mathrm{Lie}\, \mathrm{PGL}_{n,\mathcal{O}}$ and $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathcal{O})$, $\mathfrak{g}^*_{X_m} = \mathfrak{g}^* \otimes_{\mathcal{O}} \mathcal{O}_{X_m}$. We apply Lemma 2.10 to the morphisms $\pi_k : X_k \to Y_k$, with $G = \mathrm{PGL}_{n,\mathcal{O}}$, to obtain complexes (not necessarily exact) of coherent sheaves on X_k:
$$(\star_k) : 0 \to \pi_k^* \Omega_{Y_k/\mathcal{O}} \to \Omega_{X_k/\mathcal{O}} \to \mathfrak{g}^*_{X_k} \to 0.$$
We observe that e.g. $i_x^*(\star_m)$ is the complex
$$i_x^*(\star_m) : 0 \to T_y^* Y_m \to T_x^* X_m \to \mathfrak{g}^* \to 0.$$
Here we write $T_x^* X_m = i_x^* \Omega_{X_m/\mathcal{O}}$, by definition, and call it the Zariski cotangent space of X_m at the point x.

Lemma 2.11.

1. *The complex $(\star_m)[1/p]$ on $X_m[1/p]$ is an exact sequence of locally free sheaves above a Zariski open neighbourhood of the point y.*
2. *The complex $(\star_{m+1})[1/p]$ on $X_{m+1}[1/p]$ is an exact sequence of locally free sheaves above a Zariski open neighbourhood of the point $y(\gamma)$, for any $\gamma \in \Gamma$.*
3. *The complex $(\star_{m+2})[1/p]$ on $X_{m+2}[1/p]$ is an exact sequence of locally free sheaves above a Zariski open neighbourhood of the point $y(\gamma, \delta)$, for any $\gamma, \delta \in \Gamma$.*

Proof. We show that $\pi_m[1/p]$ is a $\mathrm{PGL}_{n,E}$-torsor above a Zariski open neighbourhood of y. The same proof shows the analogous statement for the points $y(\gamma)$ and $y(\gamma, \delta)$, and in each case implies the statement in the lemma (since a $\mathrm{PGL}_{n,E}$-torsor is in particular smooth). Let U denote the open subset of $X_m[1/p]$ corresponding to tuples $(u_1, \ldots, u_m)$ which generate an absolutely irreducible subgroup of GL_n. Then [Ric88, Theorem 4.1] shows that U is precisely the set of stable points of $X_m[1/p]$ (for the action of $\mathrm{PGL}_{n,E}$). In particular, $\pi_m(U)$ is an open subset of $Y_m[1/p]$ and $U = \pi_m^{-1}(\pi_m(U))$. Each point of U has a trivial stabilizer for the $\mathrm{PGL}_{n,E}$ action (Schur's lemma), so it follows from [BR85, Proposition 8.2] that $\pi_m|_U : U \to \pi_m(U)$ is a $\mathrm{PGL}_{n,E}$-torsor, as required. □

We can use the compactness of Γ to upgrade the previous lemma to the following uniform integral statement.

Lemma 2.12. *We can find an integer $k_1 \geq 0$ with the following properties:*

1. *The cohomology of the complex $i_x^*(\star_m)$ is annihilated by p^{k_1}.*
2. *For any $\gamma \in \Gamma$, the cohomology of the complex $i_{x(\gamma)}^*(\star_{m+1})$ is annihilated by p^{k_1}.*
3. *For any $\gamma, \delta \in \Gamma$, the cohomology of the complex $i_{x(\gamma,\delta)}^*(\star_{m+2})$ is annihilated by p^{k_1}.*

Proof. The first part of the lemma follows by Lemma 2.11. In fact, we can find numbers k, $k(\gamma)$, and $k(\gamma, \delta)$ such that the requirements of each point of the lemma are satisfied if k_1 is replaced by k, $k(\gamma)$, and $k(\gamma, \delta)$ in each case. What we must show is that we can find k_1 which exceeds k, $k(\gamma)$, and $k(\gamma, \delta)$ for all $\gamma, \delta \in \Gamma$.

To this end, let us suppose that k, $k(\gamma)$, and $k(\gamma, \delta)$ have been chosen to each take their smallest possible values. It suffices to show that $k(\gamma)$ and $k(\gamma, \delta)$ are locally constant as functions of $\gamma, \delta \in \Gamma$. Since Γ is compact, this will imply that they are in fact bounded. This local constancy is a consequence of the second part of Lemma 2.13 below. □

Lemma 2.13. *Let Z be a scheme of finite type over $\mathcal{O}$. If $z \in Z(\mathcal{O})$, we write $i_z : \operatorname{Spec} \mathcal{O} \to Z$ for the corresponding morphism.*

1. *Let $\mathcal{F}$ be a coherent sheaf on Z such that $\mathcal{F}[1/p]$ is locally free on a Zariski open neighbourhood V_z of $z \in Z[1/p]$. Then for any $z \in Z(\mathcal{O})$, there exists an open (for the p-adic topology) neighbourhood U of z in $Z(\mathcal{O})$ such that for any $z' \in U$, $i_{z'}^* \mathcal{F} \cong i_z^* \mathcal{F}$ as $\mathcal{O}$-modules.*
2. *Let $z \in Z(\mathcal{O})$ and let*
$$(\star) : 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
be a complex of coherent sheaves on Z, not necessarily exact, but such that on a Zariski open neighbourhood of $z \in Z[1/p]$
$$(\star[1/p]) : 0 \to \mathcal{F}_1[1/p] \to \mathcal{F}_2[1/p] \to \mathcal{F}_3[1/p] \to 0$$
is an exact sequence of locally free sheaves. Then there exists a p-adically open neighbourhood U of z in $Z(\mathcal{O})$ and an integer $N \geq 0$ such that for all $z' \in U$, $p^N H^*(i_{z'}^*(\star)) = 0$.

Proof. In each case we are free to replace Z by a Zariski open neighbourhood of the closed point specializing z. We can therefore assume that $Z = \operatorname{Spec} B$ is affine. In the first case we can assume that $\mathcal{F}$ corresponds to a finite B-module M and that there is an exact sequence
$$B^a \to B^b \to M \to 0.$$

We may assume that $\mathcal{F}[1/p]$ has constant rank $b-k$ on V_z, so we get a continuous map $Z(\mathcal{O}) \cap V_z(E) \to M_{a\times b}(\mathcal{O}) \cap M_{a\times b,k}(E)$, where $M_{a\times b,k} \subset M_{a\times b}$ is the locally closed subscheme of matrices of rank k (equivalently with vanishing $(k+1)\times(k+1)$ minors but at least one non-zero $k\times k$ minor). Note that $Z(\mathcal{O}) \cap V_z(E)$ is p-adically open in $Z(\mathcal{O})$. We are therefore reduced to showing that any matrix $T \in M_{a\times b}(\mathcal{O}) \cap M_{a\times b,k}(E)$ has an open neighbourhood U such that for $T' \in U$, $\mathcal{O}^b/T'\mathcal{O}^a$ is isomorphic to $\mathcal{O}^b/T\mathcal{O}^a$. In other words, we need to show that there is an open neighbourhood U in which the Smith normal form of T is constant. Let m be the largest valuation of a non-zero minor of T. Choosing U so that for any $T' \in U$, the minors of T' are congruent to those of T modulo ϖ^{m+1}, we see that the Smith normal forms of T and T' are indeed equal. (Note that the assumption that the E-rank is constant is necessary; otherwise we have the example of $M = \mathcal{O}[x]/x$ at the point $x = 0$, where $\mathcal{O}$ is a limit of $\mathcal{O}/(\varpi^N)$'s.)

We now turn to the second part. It suffices to show that we can find an integer $N \geq 0$ and a p-adically open neighbourhood U of z such that for all $z' \in U$, p^N annihilates $\ker(i_{z'}^*\mathcal{F}_2 \to i_{z'}^*\mathcal{F}_3)/\operatorname{im}(i_{z'}^*\mathcal{F}_1 \to i_{z'}^*\mathcal{F}_2)$. Our hypotheses imply that for z' in a Zariski open neighbourhood of z, this group is contained in the torsion subgroup of $i_{z'}^*\mathcal{F}_2/\operatorname{im}(i_{z'}^*\mathcal{F}_1 \to i_{z'}^*\mathcal{F}_2) = i_{z'}^*(\mathcal{F}_2/\operatorname{im}(\mathcal{F}_1 \to \mathcal{F}_2))$, so the result follows on applying the first part to $\mathcal{F}_2/\operatorname{im}(\mathcal{F}_1 \to \mathcal{F}_2)$. □

We are now in a position to prove Proposition 2.9. Recall that we write $A = \mathcal{O} \oplus \epsilon E/\mathcal{O}$. It is helpful to first note that if X is a scheme over $\mathcal{O}$ and $x \in X(\mathcal{O})$, then the fibre of $X(A) \to X(\mathcal{O})$ above x is canonically identified with $\operatorname{Hom}_{\mathcal{O}}(T_x^*X, E/\mathcal{O})$.

Proof of Proposition 2.9. Let the integer k_1 be as in Lemma 2.12. We will show that we can take $k_0 = 6k_1$. Taking the Pontryagin dual of $i_x^*(\star_m)$ and $i_{x(\gamma)}^*(\star_{m+1})$ gives us, for any $\gamma \in \Gamma$, a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g}\otimes_{\mathcal{O}} E/\mathcal{O} \arrow[r] & \operatorname{Hom}_{\mathcal{O}}(T_x^*X_m, E/\mathcal{O}) \arrow[r] & \operatorname{Hom}_{\mathcal{O}}(T_y^*Y_m, E/\mathcal{O}) \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{g}\otimes_{\mathcal{O}} E/\mathcal{O} \arrow[r] \arrow[u] & \operatorname{Hom}_{\mathcal{O}}(T_{x(\gamma)}^*X_{m+1}, E/\mathcal{O}) \arrow[r] \arrow[u] & \operatorname{Hom}_{\mathcal{O}}(T_{y(\gamma)}^*Y_{m+1}, E/\mathcal{O}) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

The first vertical arrow is the identity, while the other two vertical arrows correspond to forgetting the last entry in GL_n^{m+1}. Both rows have cohomology annihilated by p^{k_1}. Consequently the induced map

(2.13.1)
$$\operatorname{Hom}_{\mathcal{O}}(T_{x(\gamma)}^*X_{m+1}, E/\mathcal{O}) \to \operatorname{Hom}_{\mathcal{O}}(T_x^*X_m, E/\mathcal{O}) \times_{\operatorname{Hom}_{\mathcal{O}}(T_y^*Y_m, E/\mathcal{O})} \operatorname{Hom}_{\mathcal{O}}(T_{y(\gamma)}^*Y_{m+1}, E/\mathcal{O})$$

has kernel and cokernel annihilated by p^{2k_1}. In particular, given an element z of the target we can define an element of the source unambiguously as follows: choose a pre-image z' of $p^{2k_1}z$. Then $z'' = p^{2k_1}z'$ depends only on z and has image $p^{4k_1}z$.

Now suppose given a pseudocharacter t' over A lifting t. The data of the pseudocharacter t' (more precisely, t'_m) determines an element $y' \in \operatorname{Hom}(T_y^*Y_m, E/\mathcal{O})$. We fix a choice of $x' \in \operatorname{Hom}_{\mathcal{O}}(T_x^*X_m, E/\mathcal{O})$ with image equal to $p^{k_1}y'$. This corresponds to a tuple of elements $(g'_1, \ldots, g'_m) \in \mathrm{GL}_n(A)^m$ lifting the element $(g_1, \ldots, g_m) = (\rho(\gamma_1), \ldots, \rho(\gamma_m))$. If x'' is any other choice of element with image equal to $p^{k_1}y'$, then $p^{k_1}x - p^{k_1}x''$ is in the image of $\mathfrak{g}\otimes_{\mathcal{O}} E/\mathcal{O}$.

The pseudocharacter t' also determines elements

$$y'(\gamma) \in \mathrm{Hom}(T^*_{y(\gamma)} Y_{m+1}, E/\mathcal{O})$$

for any $\gamma \in \Gamma$, and the pair $(p^{k_1} x', p^{2k_1} y'(\gamma))$ lies on the right-hand side of (2.13.1). We may define $\rho'(\gamma)$ uniquely as follows: it is the lift of $p^{4k_1}(p^{k_1} x', p^{2k_1} y'(\gamma))$ associated to the map (2.13.1). Then $\rho' : \Gamma \to \mathrm{GL}_n(A)$ has associated trace function $\operatorname{tr} \rho' = \alpha_{6k_1} \circ t'$, and its conjugacy class under $1 + \epsilon \mathfrak{g} \otimes_{\mathcal{O}} E/\mathcal{O}$ is independent of choices. We can verify that ρ' is a homomorphism (i.e. that it respects multiplication) using the fact that t' is a pseudocharacter, together with a diagram with rows corresponding to elements $x(\gamma\delta)$ and $x(\gamma, \delta)$.

Now suppose given two liftings ρ'_1, ρ'_2 of ρ to A with $\operatorname{tr} \rho'_1 = \operatorname{tr} \rho'_2 = t'$, say. This implies that for each $\gamma \in \Gamma$, the tuples $(\rho'_i(\gamma_1), \dots, \rho'_i(\gamma_m), \rho'_i(\gamma))$ $(i = 1, 2)$, when identified with elements of $\mathrm{Hom}_{\mathcal{O}}(T^*_{x(\gamma)} X_{m+1}, E/\mathcal{O})$, have equal image in $\mathrm{Hom}_{\mathcal{O}}(T^*_{y(\gamma)} Y_{m+1}, E/\mathcal{O})$. Consequently there is X_γ in $\mathfrak{g} \otimes_{\mathcal{O}} E/\mathcal{O}$ taking α_{k_1} of the first tuple to α_{k_1} of the second. Passing to the top row of the commutative diagram, we see that for any $\gamma, \gamma' \in \Gamma$, we have $p^{k_1}(X_\gamma - X_{\gamma'}) = 0$, hence $X = p^{k_1} X_\gamma$ is independent of $\gamma \in \Gamma$. It follows that X takes $\alpha_{2k_1} \circ \rho'_1$ to $\alpha_{2k_1} \circ \rho'_2$.

It remains to verify that if t' is continuous, then so is ρ'. It is enough to show that for each $s \geq 1$, there exists an open subgroup $N \subset \Gamma$ such that $\rho'(N)$ takes values in the subgroup $1 + \varpi^s M_n(\mathcal{O})$ of $\mathrm{GL}_n(A)$. Since $\mathbf{Z}[\mathrm{GL}_n^{m+1}]^{\mathrm{GL}_n}$ is a finitely generated $\mathbf{Z}$-algebra and Γ is compact, there exists $r \geq 1$ such that t'_{m+1} takes values in $\mathrm{Map}(\Gamma^{m+1}, A_r)$, where $A_r = \mathcal{O} \oplus \epsilon \varpi^{-r} \mathcal{O}/\mathcal{O} \subset A$. Increasing s, we can assume that $s \geq r$, that $\rho'(\gamma_1), \dots, \rho'(\gamma_m)$ lie in $\mathrm{GL}_n(A_s)$, and that there exists an open subgroup $N \subset \Gamma$ such that for all $\gamma \in N$, $\rho(\gamma) \in 1 + \varpi^s M_n(\mathcal{O})$ and $t'_{m+1}(\gamma_1, \dots, \gamma_m, \gamma) \equiv t'_{m+1}(\gamma_1, \dots, \gamma_m, 1) \bmod \varpi^s A_s$. We observe that for $\gamma \in N$, there is a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}}(T^*_{x(\gamma)} X_{m+1}, \varpi^{-s}\mathcal{O}/\mathcal{O}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{O}}(T^*_x X_m, \varpi^{-s}\mathcal{O}/\mathcal{O}) \times \mathrm{Hom}_{\mathcal{O}}(T^*_{y(\gamma)} Y_{m+1}, \varpi^{-s}\mathcal{O}/\mathcal{O}) \arrow[d] \\
\mathrm{Hom}_{\mathcal{O}}(T^*_{x(1)} X_{m+1}, \varpi^{-s}\mathcal{O}/\mathcal{O}) \arrow[r] & \mathrm{Hom}_{\mathcal{O}}(T^*_x X_m, \varpi^{-s}\mathcal{O}/\mathcal{O}) \times \mathrm{Hom}_{\mathcal{O}}(T^*_{y(1)} Y_{m+1}, \varpi^{-s}\mathcal{O}/\mathcal{O})
\end{tikzcd}$$

where the horizontal arrows are the ones already considered in (2.13.1) (suppressing the subscripts indicating the fibre product to save space), and the vertical ones are bijections arising from the identification between $\mathrm{Hom}_{\mathcal{O}}(T^*_{x(\gamma)} X_{m+1}, \varpi^{-s}\mathcal{O}/\mathcal{O})$ and the fibre of $X_{m+1}(\mathcal{O}/\varpi^s \oplus \epsilon \varpi^{-s}\mathcal{O}/\mathcal{O}) \to X_{m+1}(\mathcal{O}/\varpi^s)$ above $x(\gamma) \bmod \varpi^s = x(1) \bmod \varpi^s$. The horizontal arrows have kernels annihilated by p^{2k_1}. Our assumptions imply that the elements of

$$\mathrm{Hom}_{\mathcal{O}}(T^*_{x(\gamma)} X_{m+1}, \varpi^{-s}\mathcal{O}/\mathcal{O})$$

and

$$\mathrm{Hom}_{\mathcal{O}}(T^*_{x(1)} X_{m+1}, \varpi^{-s}\mathcal{O}/\mathcal{O})$$

corresponding to the images of $\rho'(\gamma)$ and $\rho'(1)$ in $\mathrm{GL}_n(\mathcal{O}/\varpi^s \oplus \epsilon\varpi^{-s}\mathcal{O}/\mathcal{O})$ are identified under the above bijection. This is what we needed to prove. $\square$

2.14. **Pseudocharacters – Galois deformation theory.** We again fix a prime p and a finite extension $E/\mathbf{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k.

Let F be a number field, and let S be a finite set of finite places of F, containing the p-adic ones. Let $\overline{\rho} : G_{F,S} \to \mathrm{GL}_n(k)$ be a continuous representation. Let $\overline{D}$ denote the associated group determinant over k.

We write $\mathrm{Def}_{\overline{D},S} : \mathcal{C}_{\mathcal{O}} \to \mathrm{Sets}$ for the functor which associates to each $A \in \mathcal{C}_{\mathcal{O}}$ the set of continuous group determinants D of $G_{F,S}$ over A such that $D \bmod \mathfrak{m}_A = \overline{D}$.

Proposition 2.15. *The functor* $\mathrm{Def}_{\overline{D},S}$ *is represented by an object* $R_{\overline{D},S} \in \mathcal{C}_{\mathcal{O}}$.

Proof. See [Che14, §3.3]. (This reference deals with the case $\mathcal{O} = W(k)$, but the extension to general coefficients is trivial.) $\square$

Lemma 2.16. *Fix an integer* $q \geq 0$. *Then there exists an integer* $g_0 = g_0(S, \overline{D}, q)$ *such that for any set* Q *of finite places of* F *such that* $|Q| \leq q$, *there exists a surjection* $\mathcal{O}[\![X_1, \dots, X_{g_0}]\!] \to R_{\overline{D},S\cup Q}$.

Proof. Let L/F denote the extension cut out by $\overline{\rho}$, and let $M_{S\cup Q}$ denote the maximal pro-p extension of L unramified outside $S \cup Q$. Then there exists $g_1 = g_1(S, \overline{\rho}, q)$ such that the group $\mathrm{Gal}(M_{S\cup Q}/F)$ can be topologically generated by g_1 elements (because the dimension of the space $H^1(G_{L,S\cup Q}, k)$ can be bounded in a way depending only on q). Moreover, any deformation of $\overline{D}$ to $G_{F,S\cup Q}$ in fact factors through $\mathrm{Gal}(M_{S\cup Q}/F)$ (by [Che14, Lemma 3.8]). The statement of the lemma follows on applying e.g. the results of [Che14, §2.37]. $\square$

Now fix integers $a \leq b$, and let $\mathcal{E}^{[a,b]}_{F,S}$ denote the category of finite cardinality $\mathbf{Z}_p[G_{F,S}]$-modules M such that for each place $v|p$ of F, M is isomorphic as $\mathbf{Z}_p[G_{F_v}]$-module to a subquotient of lattice in a semistable representation of G_{F_v} with Hodge–Tate weights in $[a, b]$. This defines a stable condition in the sense of [WWE19, Definition 2.3.1].

We write $\mathrm{Def}^{[a,b]}_{\overline{D},S} \subset \mathrm{Def}_{\overline{D},S}$ for the subfunctor which assigns to $A \in \mathcal{C}_{\mathcal{O}}$ the set of determinants D over A satisfying the following condition:

(2.16.1) There exists a Cayley–Hamilton representation $(\mathcal{O}[G_{F,S}], D) \to (B, D')$ over A such that for each $n \geq 1$, $B/\mathfrak{m}_A^n B$ lies in $\mathcal{E}^{[a,b]}_{F,S}$, when equipped with its left $G_{F,S}$-action.

The notion of Cayley–Hamilton representation is defined in [WWE19, Definition 2.1.8]. We recall that it is an $\mathcal{O}$-algebra homomorphism $\rho : \mathcal{O}[G_{F,S}] \to B$, where B is a finitely generated A-algebra and $D' : B \to A$ is a determinant such that $D' \circ \rho = D$ and the associated Cayley–Hamilton ideal $\mathrm{CH}(D') \subset B$ is zero. Note that in this situation B is necessarily finitely generated as an A-module ([WWE19, Proposition 2.1.7]).

Proposition 2.17. *The functor* $\mathrm{Def}^{[a,b]}_{\overline{D},S}$ *is represented by an object* $R^{[a,b]}_{\overline{D},S} \in \mathcal{C}_{\mathcal{O}}$.

Proof. See [WWE19, Theorem 2.5.5]. $\square$

Now suppose given a lift $\rho : G_{F,S} \to \mathrm{GL}_n(\mathcal{O})$ of $\overline{\rho}$ with the following properties:

- $\rho \otimes_{\mathcal{O}} E$ is absolutely irreducible.
- For each place $v|p$ of F, $\rho|_{G_{F_v}} \otimes_{\mathcal{O}} E$ is semistable with Hodge–Tate weights in the interval $[a, b]$.

Let D denote the associated group determinant over $\mathcal{O}$. Then D determines a homomorphism $R^{[a,b]}_{\overline{D},S} \to \mathcal{O}$. We write $\mathfrak{q}$ for the kernel. Let $W = \operatorname{ad}\rho$, $W_E = W \otimes_{\mathcal{O}} E$, $W_{E/\mathcal{O}} = W_E/W$, $W_m = \operatorname{ad}\rho \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$. We have a Selmer group $H^1_{\mathcal{E}^{[a,b]}_{F,S}}(F, W_m)$ defined by local conditions: if $v \notin S$, we take the unramified subgroup of $H^1(F_v, W_m)$, if $v \in S - S_p$ we impose no condition and if $v \in S_p$ we take the subspace of $H^1(F_v, W_m)$ corresponding to self-extensions of $\rho|_{G_{F_v}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$ which are subquotients of lattices in semistable representations with Hodge–Tate weights in the interval $[a, b]$.

Proposition 2.18. *There exists a canonical homomorphism*

$$\operatorname{tr}_m : H^1_{\mathcal{E}^{[a,b]}_{F,S}}(F, W_m) \to \operatorname{Hom}_{\mathcal{O}}(\mathfrak{q}/\mathfrak{q}^2, \mathcal{O}/\varpi^m). \tag{2.18.1}$$

Moreover, there is a constant $c \geq 1$ depending only on ρ (and not on S, $[a,b]$, or m) such that for any $m \geq 1$, the kernel and cokernel of tr_m are both annihilated by p^c.

In applications of the proposition we will enlarge S by adding Taylor–Wiles places. This is why it is important that the constant c is independent of the set S.

Proof. We first describe the map tr_m. Let $A_m = \mathcal{O} \oplus \epsilon\varpi^{-m}\mathcal{O}/\mathcal{O} \subset A = \mathcal{O} \oplus \epsilon E/\mathcal{O}$. If $k \geq 0$, then we write $\alpha_k : A_m \to A_m$ for the $\mathcal{O}$-algebra homomorphism that sends ϵ to $p^k\epsilon$. A class $[\phi] \in H^1_{\mathcal{E}^{[a,b]}_{F,S}}(F, W_m)$ corresponds to an equivalence class of liftings

$$\rho_\phi : G_{F,S} \to \mathrm{GL}_n(A_m)$$

such that $\rho_\phi \bmod (\epsilon) = \rho$ and for each $N \geq 1$, $\rho_\phi \bmod \varpi^N \in \mathcal{E}^{[a,b]}_{F,S}$ (this can be seen by considering the extension of scalars along the injective ring homomorphism $A_m \hookrightarrow \mathcal{O} \times \mathcal{O}/\varpi^m[\epsilon]$ and using the fact that $\mathcal{E}^{[a,b]}_{F,S}$ is closed under taking sub-$\mathbf{Z}_p[G_{F,S}]$-modules). On the other hand, we can identify $\operatorname{Hom}_{\mathcal{O}}(\mathfrak{q}/\mathfrak{q}^2, \mathcal{O}/\varpi^m)$ with the pre-image under the map

$$\operatorname{Hom}_{\mathcal{O}}(R^{[a,b]}_{\overline{D},S}, A_m) \to \operatorname{Hom}_{\mathcal{O}}(R^{[a,b]}_{\overline{D},S}, \mathcal{O})$$

of the classifying map of D. The map tr_m is the one which sends $[\phi]$ to the classifying map of the pseudocharacter $\operatorname{tr}\rho_\phi$ over A_m. Note that multiplication by p^k on either side of (2.18.1) corresponds at the level of representations (resp. determinants) to composition with α_k.

Next we analyze the kernel of tr_m. Suppose that $\operatorname{tr}\rho_\phi = \operatorname{tr}\rho$. Let k_0 be the constant of Proposition 2.9: it depends only on $\rho(G_{F,S}) \subset \mathrm{GL}_n(\mathcal{O})$. Then we can find $X \in M_n(E/\mathcal{O})$ such that

$$(1 + \epsilon X)\rho_{p^{k_0}\phi}(1 - \epsilon X) = \rho,$$

or equivalently such that $p^{k_0}\phi$ becomes a coboundary in $H^1(F, W_{E/\mathcal{O}})$. Since the kernel of $H^1(F, W_m) \to H^1(F, W_{E/\mathcal{O}})$ is isomorphic to $H^0(F, W_{E/\mathcal{O}}) \otimes \mathcal{O}/\varpi^m$, it is killed by a uniformly bounded power of p and we see that the same is true for the kernel of tr_m.

Now we analyze the cokernel of tr_m. This is more subtle. Let D' be a determinant of $G_{F,S}$ over A_m corresponding to an element of the right-hand side of (2.18.1). By assumption, there exists a Cayley–Hamilton representation $r : A_m[G_{F,S}] \to B$ and a determinant $D'' : B \to A_m$ such that $D' = D'' \circ r$, and the finite quotients of B lie in $\mathcal{E}^{[a,b]}_{F,S}$. We may assume without loss of generality that r is surjective. According to the characterization of $\ker(D') \subset A_m[G_{F,S}]$ given by [Che14, Lemma 1.19], we have $\ker(r) \subset \ker(D')$.

By Proposition 2.9, there exists a homomorphism $\rho_\phi : G_{F,S} \to \mathrm{GL}_n(A)$ such that the associated group determinant is $\alpha_{k_0} \circ D'$. It also follows from Proposition 2.9 that the associated cohomology class $[\phi] \in H^1(F, W_{E/\mathcal{O}})$ is killed by multiplication by $\varpi^m p^{k_0}$ (as $\mathrm{tr}(\rho_{\varpi^m \phi}) = \mathrm{tr}(\rho)$). We deduce that $p^{k_0}[\phi]$ is contained in the image of $H^1(F, W_m)$ in $H^1(F, W_{E/\mathcal{O}})$. So we may in fact assume that we have $\rho_\phi : G_{F,S} \to \mathrm{GL}_n(A_m)$ such that the associated group determinant is $\alpha_{2k_0} \circ D'$.

We must show that there is $c \geq 0$ depending only on ρ such that for each $N \geq 1$, $\alpha_c \circ \rho_\phi \bmod \varpi^N$ defines an object of $\mathcal{E}_{F,S}^{[a,b]}$ (as then $p^c[\phi]$ is a pre-image under (2.18.1) of $\alpha_{2k_0+c} \circ D'$).

Let $\mathcal{A}_\phi = \rho_\phi(A_m[G_{F,S}]) \subset M_n(A_m)$. Let $k_1 \geq 0$ be an integer such that $p^{k_1} M_n(\mathcal{O}) \subset \rho(\mathcal{O}[G_{F,S}])$ (this exists since $\rho \otimes_{\mathcal{O}} E$ is absolutely irreducible, by assumption, and hence $\rho(E[G_{F,S}]) = M_n(E)$). Then $\mathcal{A}_{p^{k_1}\phi}$ contains $p^{k_1} M_n(A_m)$. Indeed, let E_{ij} denote the elementary matrix in $M_n(\mathcal{O})$ with entries 1 in the (i,j) spot and 0 elsewhere. Then $p^{k_1} E_{ij} \in \rho(\mathcal{O}[G_{F,S}])$, so there is $X_{ij} \in M_n(\mathcal{O}/\varpi^m)$ such that $p^{k_1} E_{ij} + \epsilon X_{ij} \in \mathcal{A}_\phi$ and $p^{k_1} E_{ij} + \epsilon p^{k_1} X_{ij} \in \mathcal{A}_{p^{k_1}\phi}$. After multiplying by ϵ, we see that $p^{k_1} \epsilon E_{ij} \in \mathcal{A}_{p^{k_1}\phi}$ for all (i,j), hence $p^{k_1} E_{ij} \in \mathcal{A}_{p^{k_1}\phi}$.

Let $D''' : \mathcal{A}_{p^{k_1}\phi} \to A_m$ denote the determinant induced by the natural inclusion $\mathcal{A}_{p^{k_1}\phi} \to M_n(A_m)$. If $x \in \ker D'''$, then [Che14, Lemma 1.19] shows that $\mathrm{tr}(x p^{k_1} E_{ij}) = 0$ for all i, j. (Here $\mathrm{tr} : M_n(A_m) \to A_m$ is the usual trace of an $n \times n$ matrix, not a pseudocharacter.) Hence $\ker D'''$ is contained in the intersection of $\mathcal{A}_{p^{k_1}\phi}$ with $M_n(A_m)[p^{k_1}]$, and is therefore annihilated by the homomorphism $\alpha_{k_1} : M_n(A_m) \to M_n(A_m)$.

It follows that there exists a commutative diagram of A_m-algebras

$$\begin{tikzcd}
A_m[G_{F,S}] \arrow[r, "\rho_{p^{k_1}\phi}"] \arrow[d, "r"] & \mathcal{A}_{p^{k_1}\phi} \arrow[r] \arrow[d] & M_n(A_m) \arrow[d, "\alpha_{k_1}"] \\
B \arrow[r] & A_m[G_{F,S}]/\ker(\alpha_{2k_0+k_1} \circ D') \arrow[r] & M_n(A_m).
\end{tikzcd}$$

Indeed, the quotient map $A_m[G_{F,S}] \to A_m[G_{F,S}]/\ker(\alpha_{2k_0+k_1} \circ D')$ factors through B because $\ker(r) \subset \ker(D') \subset \ker(\alpha_{2k_0+k_1} \circ D')$, and bottom right arrow exists because $\alpha_{2k_0+k_1} \circ D' = D''' \circ \rho_{p^{k_1}\phi}$. The existence of this diagram shows that the finite quotients of $M_n(A_m)$, with induced action of $A_m[G_{F,S}]$ by left multiplication via $\rho_{p^{2k_1}\phi}$, satisfy condition $\mathcal{E}_{F,S}^{[a,b]}$ (since this holds for finite quotients of B). This implies that the finite quotients of $\alpha_{2k_1} \circ \rho_\phi$ also satisfy the condition $\mathcal{E}_{F,S}^{[a,b]}$. We deduce that the cokernel of tr_m is annihilated by $p^{2k_0+2k_1}$. $\square$

2.19. **Pseudocharacters – Taylor–Wiles data.** We continue our discussion of the Galois deformation theory of pseudocharacters, now focusing on what happens when we impose conjugate self-duality conditions and allow additional primes of ramification. We thus fix the following notation:

- p is a prime and $E/\mathbf{Q}_p$ is a coefficient field.
- F/F^+ is a CM quadratic extension of a totally real field.

- S is a finite set of finite places of F^+ containing the p-adic ones. We assume that each place of S splits in F, and fix for each $v \in S$ a choice of place $\widetilde{v}$ of F lying above v. We set $S_p = \{v \mid v|p\}$ and $\widetilde{S} = \{\widetilde{v} \mid v \in S\}$.
- $r : G_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$ is a continuous representation such that $r|_{G_{F,S}} \otimes_{\mathcal{O}} E$ is absolutely irreducible. We set $\rho = r|_{G_{F,S}} : G_{F,S} \to \mathrm{GL}_n(\mathcal{O})$ and write D for the group determinant of ρ and $\overline{D}$ for its reduction modulo ϖ. We set $\chi = \nu \circ r$. The existence of r implies that the $\mathcal{O}[G_{F,S}]$-module structure on $W = \operatorname{ad} \rho$ extends to an $\mathcal{O}[G_{F^+,S}]$-module structure (set $W = \operatorname{ad} r$, where $\operatorname{ad} r$ is defined as in [CHT08, §2.1]), and similarly for W_E, W_m, etc.
- $a \leq b$ are integers such that D defines a homomorphism $R^{[a,b]}_{\overline{D},S} \to \mathcal{O}$. Thus for each $v \in S_p$, $\rho|_{G_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} E$ is semistable with all Hodge–Tate weights in the range $[a, b]$. Note that $\chi|_{G_{F_{\widetilde{v}}}}$ is then semistable and there exists $w \in \mathbf{Z}$ such that $\chi\epsilon^w$ has finite order. We assume moreover that $a + b = w$.

Let R_S denote the quotient of $R^{[a,b]}_{\overline{D},S}$ corresponding to pseudocharacters D' such that $(D')^c = (D')^\vee \otimes \chi|_{G_{F,S}}$. Then ρ determines a homomorphism $R_S \to \mathcal{O}$, and we write $\mathfrak{q}_S$ for its kernel.

We define Selmer conditions $\mathcal{L}_S = \{\mathcal{L}_v\} = \{\mathcal{L}_{v,m}\}$ for W_m as follows: if $v \notin S$, then $\mathcal{L}_v$ is the unramified subgroup of $H^1(F_v^+, W_m)$. If $v \in S - S_p$, then $\mathcal{L}_v = H^1(F_v^+, W_m)$. If $v \in S_p$, then $\mathcal{L}_v$ is the subspace of $H^1(F_v^+, W_m)$ corresponding to self-extensions of $\rho|_{G_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$ which are subquotients of lattices in semistable representations with Hodge–Tate weights in the interval $[a, b]$. We define dual Selmer conditions $\mathcal{L}^\perp = \{\mathcal{L}_v^\perp\}$ to be given by the annihilators of $\mathcal{L}_v$ under local duality.

The corresponding Selmer groups are defined by

$$H^1_{\mathcal{L}_S}(F^+, W_m) = \ker\left(H^1(F^+, W_m) \to \prod_v H^1(F_v^+, W_m)/\mathcal{L}_{v,m}\right)$$
$$H^1_{\mathcal{L}_S^\perp}(F^+, W_m(1)) = \ker\left(H^1(F^+, W_m(1)) \to \prod_v H^1(F_v^+, W_m(1))/\mathcal{L}_{v,m}^\perp\right).$$

These Selmer groups are finite length $\mathcal{O}$-modules. We denote their length by $h^1_{\mathcal{L}_S}(F^+, W_m), h^1_{\mathcal{L}_S^\perp}(F^+, W_m(1))$. Taking inverse limits with respect to the projection maps $W_{m+1} \to W_m$ and direct limits with respect to the injections $W_m \cong \varpi W_{m+1} \subset W_{m+1}$, we also define Selmer groups with characteristic 0 and divisible coefficients:

$$H^1_{\mathcal{L}_S}(F^+, W_E) = \left(\varprojlim_m H^1_{\mathcal{L}_S}(F^+, W_m)\right) \otimes_{\mathcal{O}} E$$
$$H^1_{\mathcal{L}_S}(F^+, W_{E/\mathcal{O}}) = \varinjlim_m H^1_{\mathcal{L}_S}(F^+, W_m).$$

Proposition 2.18 has the following consequence:

Proposition 2.20. *For each $m \geq 1$, there is a canonical homomorphism*

$$\mathrm{tr}_{m,S} : H^1_{\mathcal{L}_S}(F^+, W_m) \to \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_S/\mathfrak{q}_S^2, \mathcal{O}/\varpi^m). \tag{2.20.1}$$

Moreover, there is a constant $d \geq 0$ depending only on r (and not on S, $[a, b]$ or m) such that p^d annihilates the kernel and cokernel of $\mathrm{tr}_{m,S}$.

Proof. The map (2.18.1) is $\mathrm{Gal}(F/F^+)$-equivariant for the action on the left-hand side induced by the $G_{F^+,S}$ action on W_m and the action on the right-hand side defined as follows: the non-trivial element $c \in \mathrm{Gal}(F/F^+)$ acts on $R^{[a,b]}_{\overline{D},S}$ by sending a pseudocharacter D' to $(D')^{c,\vee} \otimes \chi|_{G_{F,S}}$, and this induces an action on the right-hand side of (2.18.1). Note that this action makes sense because of our condition $a+b=w$. The right-hand side of (2.20.1) is the c-invariants in the right-hand side of (2.18.1). The left-hand side of (2.20.1) maps to the c-invariants in the left-hand side of (2.18.1) with bounded kernel and cokernel. This is enough. □

Here is a variant that will be useful when it comes to deduce our main vanishing result.

Proposition 2.21. *(1) There is an isomorphism* $\mathrm{tr}_{E,S} : H^1_{\mathcal{L}_S}(F^+, W_E) \to \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_S/\mathfrak{q}_S^2, E)$.

(2) The natural map $H^1_{\mathcal{L}_S}(F^+, W_E) \to H^1(F_S/F^+, W_E)$ *identifies* $H^1_{\mathcal{L}_S}(F^+, W_E)$ *with the geometric Selmer group* $H^1_{g,S}(F^+, W_E)$

(3) Suppose that for each $v \in S$, $\rho|_{G_{F_{\tilde{v}}}}$ *is generic. Then* $H^1_{g,S}(F^+, W_E) = H^1_f(F^+, W_E)$.

Proof. The first part follows from Proposition 2.20 by taking the inverse limit over m and inverting p. The main result of [Liu07] implies that $H^1_{\mathcal{L}_S}(F^+, W_E)$ classifies (polarized) semistable self-extensions of ρ_E. A de Rham self-extension of ρ_E is automatically semistable [Nek93, Corollary 1.27], so the second part follows. The third part follows from the equality of the respective local Selmer groups in the generic case. □

For $m' \geq m$ the inverse image of $\mathcal{L}_{v,m'}$ in $H^1(F_v^+, W_m)$ under the map

$$H^1(F_v^+, W_m) \to H^1(F_v^+, W_{m'})$$

induced by the injection $W_m \to W_{m'}$ equals $\mathcal{L}_{v,m}$. Indeed, the natural map $H^1(F_v^+, W_m) \to H^1(F_v^+, W_{m'})$ corresponds to pushing forward a $\mathrm{GL}_n(A_m)$-valued lifting to $\mathrm{GL}_n(A_{m'})$, which preserves semistability (cf. the argument of [Ram93, Proposition 1.1] – here are are writing $A_m = \mathcal{O} \oplus \epsilon\varpi^{-m}\mathcal{O}/\mathcal{O}$, as in the proof of Proposition 2.18). We record a consequence of this in the following lemma.

Lemma 2.22. *The natural map* $H^1_{\mathcal{L}_S}(F^+, W_m) \to H^1_{\mathcal{L}_S}(F^+, W_{E/\mathcal{O}})[\varpi^m]$ *is surjective.*

Proof. We consider the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathcal{L}_S}(F^+, W_m) \arrow[r] \arrow[d] & H^1(F_S/F^+, W_m) \arrow[r] \arrow[d] & \bigoplus_{v \in S_p} H^1(F_v^+, W_m)/\mathcal{L}_{v,m} \arrow[d] \\ 0 \arrow[r] & H^1_{\mathcal{L}_S}(F^+, W_{E/\mathcal{O}})[\varpi^m] \arrow[r] & H^1(F_S/F^+, W_{E/\mathcal{O}})[\varpi^m] \arrow[r] & \bigoplus_{v \in S_p} \varinjlim_{m'} H^1(F_v^+, W_{m'})/\mathcal{L}_{v,m'} \end{tikzcd}$$

The central vertical map is surjective. The right vertical map is injective (by the observation preceding this lemma). So the left vertical map is surjective. □

If Q is a set of finite places of F^+ and N is a positive integer, we say that Q is *a set of Taylor–Wiles places of level* N (relative to r, S) if it satisfies the following conditions:

- $Q \cap S = \emptyset$.
- For each $v \in Q$, $v = ww^c$ splits in F; and $\rho(\mathrm{Frob}_w)$ has n distinct eigenvalues $\alpha_{w,1}, \ldots, \alpha_{w,n} \in \mathcal{O}$.

- For each $v \in Q$, $q_v \equiv 1 \bmod p^N$.

A *Taylor–Wiles datum of level* $N \geq 1$ is a tuple $(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \ldots, \alpha_{\widetilde{v},n})_{\widetilde{v} \in \widetilde{Q}})$, where Q is a set of Taylor–Wiles places of level N, $\widetilde{Q}$ is a set consisting of a choice, for each $v \in Q$, of a place $\widetilde{v}$ of F lying above v, and $(\alpha_{\widetilde{v},1}, \ldots, \alpha_{\widetilde{v},n})$ is a choice of ordering of the eigenvalues of $\rho(\mathrm{Frob}_{\widetilde{v}})$.

Lemma 2.23. *Suppose that the following conditions are satisfied:*

(1) For each $v \in S$, $\rho|_{G_{F_{\widetilde{v}}}}$ is generic.
(2) For each place $v|\infty$, $\chi(c_v) = -1$.

Then there exists $d \geq 0$ with the following property: for every $N \geq 1$, every Taylor–Wiles datum $(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \ldots, \alpha_{\widetilde{v},n})_{\widetilde{v} \in \widetilde{Q}})$ of level N, and every $1 \leq m \leq N$, we have:

$$h^1_{\mathcal{L}_{S \cup Q}}(F^+, W_m) \leq d + h^1_{\mathcal{L}^\perp_{S \cup Q}}(F^+, W_m(1)) + mn|Q| + \sum_{v \in Q} \sum_{i \neq j} \mathrm{ord}_\varpi(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j}).$$

Proof. Fix a Taylor–Wiles datum. By the usual Greenberg–Wiles formula, we have

$$h^1_{\mathcal{L}_{S \cup Q}}(F^+, W_m) = h^1_{\mathcal{L}^\perp_{S \cup Q}}(F^+, W_m(1)) + h^0(F^+, W_m) - h^0(F^+, W_m(1)) + \sum_{v \in S \cup Q} (l_{v,m} - h^0(F_{\widetilde{v}}, W_m)) - \sum_{v|\infty} l((1 + c_v) W_m),$$

where $l_{v,m} = l(\mathcal{L}_{v,m})$ and l denotes the length of an $\mathcal{O}$-module. The contribution from the infinite places is $m[F^+ : \mathbf{Q}]n(n-1)/2$, up to a uniformly bounded error. The global terms $h^0(F^+, W_m)$ and $h^0(F^+, W_m(1))$ are both uniformly bounded, the first since ρ is absolutely irreducible and the second since ρ is absolutely irreducible and $\rho, \rho(1)$ have different sets of Hodge–Tate weights.

If $v \in Q$, then $(l_{v,m} - h^0(F_{\widetilde{v}}, W_m)) = h^0(F_{\widetilde{v}}, W_m(1)) = h^0(F_{\widetilde{v}}, W_m)$, since we are assuming $m \leq N$, and this is bounded above by $nm + \sum_{i \neq j} \mathrm{ord}_\varpi(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j})$. If $v \in S - S_p$, then $(l_{v,m} - h^0(F_{\widetilde{v}}, W_m))$ is uniformly bounded, by [All16, Proposition 1.2.2].

Finally, suppose that $v \in S_p$. Let $R_v^{\square,[a,b]} \in \mathcal{C}_\mathcal{O}$ denote the object representing the functor of lifts of $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ whose projections to Artinian quotients are subquotients of lattices in semistable representations with all Hodge–Tate weights in the interval $[a,b]$. The representation $\rho|_{G_{F_{\widetilde{v}}}}$ determines a homomorphism $R_v^{\square,[a,b]} \to \mathcal{O}$. If $\mathfrak{q}_v$ denotes its kernel, then by definition we have $l_{v,m} - h^0(F_{\widetilde{v}}, W_m) = l(\mathfrak{q}_v/\mathfrak{q}_v^2 \otimes_\mathcal{O} \mathcal{O}/\varpi^m) - mn^2$. We wish to show that $l_{v,m} - h^0(F_{\widetilde{v}}, W_m) - m[F_v^+ : \mathbf{Q}_p]n(n-1)/2$ is bounded independently of m. This in turn will follow if we can show that $\mathfrak{q}_v/\mathfrak{q}_v^2 \otimes_\mathcal{O} E$ has dimension $n^2 + [F_v^+ : \mathbf{Q}_p]n(n-1)/2$ as E-vector space.

However, the argument of [Kis09, Proposition 2.3.5], together with [Liu07, Conjecture 1.0.1] (stated as a conjecture but proved in that paper), shows that the completed local ring of $R_v^{\square,[a,b]}$ at $\mathfrak{q}_v$ represents the functor $\mathcal{C}_E \to$ Sets of lifts of $\rho|_{G_{F_{\widetilde{v}}}} \otimes_\mathcal{O} E$ whose Artinian quotients are semistable with all Hodge–Tate weights in the interval $[a,b]$. The tangent space to this functor (which is equal to $\mathfrak{q}_v/\mathfrak{q}_v^2 \otimes_\mathcal{O} E$) is computed in the proof of [All16, Theorem 1.2.7], which gives the desired result. $\square$

Lemma 2.24. *Suppose M is a finitely generated $\mathcal{O}$-module and let $N \geq 1$ and $d, g \geq 0$ be integers. Suppose we know that for all $m \leq N$ we have*

$$l(M/\varpi^m) \leq gm + d.$$

Then there is a map $\mathcal{O}^g \to M/\varpi^N$ with cokernel of length $\leq d$.

Proof. We prove the lemma by induction on the number of generators of M. The lemma is obvious if M is cyclic. For a general M, we can first replace M by M/ϖ^N without changing anything. Now let $M' = M/C$ where C is a cyclic submodule of M of maximal length and let $N' \leq N$ be the maximal length of a cyclic submodule of M'. It suffices to prove that there is a map $\mathcal{O}^{g-1} \to M'/\varpi^{N'} = M'/\varpi^N$ with cokernel of length $\leq d$. For all $m \leq N'$ we have $l(M'/\varpi^m) = l(M/\varpi^m) - m \leq (g-1)m + d$. By induction, we have the desired map $\mathcal{O}^{g-1} \to M'/\varpi^{N'}$. □

Corollary 2.25. *Suppose that ρ satisfies the hypotheses of Lemma 2.23. Then there exists $d \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ and every Taylor–Wiles datum of level N, there is a map*

$$\mathcal{O}^{n|Q|} \to H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_N)$$

with cokernel of length $\leq d + h^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_N(1)) + \sum_{v\in Q}\sum_{i\neq j} \mathrm{ord}_\varpi(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j})$.

Proof. Using Lemma 2.23 and Lemma 2.24, we see that it is enough to find $d_0, d_1 \in \mathbf{N}$ such that for any $1 \leq m \leq N$ and any Taylor–Wiles datum of level N, we have

$$l(H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_N)/(\varpi^m)) \leq h^1_{\mathcal{L}_{S\cup Q}}(F^+, W_m) + d_0 \tag{2.25.1}$$

and

$$h^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_m(1)) \leq h^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_N(1)) + d_1. \tag{2.25.2}$$

We treat these in turn. For the first inequality, we note that Lemma 2.22 shows that the map

$$H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_m) \to H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^m]$$

is surjective, with kernel a subquotient of $H^0(F^+, W_{E/\mathcal{O}})$. It follows that there is a surjective homomorphism

$$H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_N)/(\varpi^m) \to H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^N]/(\varpi^m)$$

with kernel a subquotient of $H^0(F^+, W_{E/\mathcal{O}})$. Since we have

$$l(H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^N]/(\varpi^m)) = l(H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^m]),$$

we see that (2.25.1) holds with $d_0 = h^0(F^+, W_{E/\mathcal{O}})$.

For the second inequality, we note that the kernel of the natural map

$$H^1(F_S/F^+, W_m(1)) \to H^1(F_S/F^+, W_N(1)) \tag{2.25.3}$$

is contained in the kernel of the map

$$H^1(F_S/F^+, W_m(1)) \to H^1(F_S/F^+, W_{E/\mathcal{O}}(1)),$$

which is a subquotient of $H^0(F^+, W_{E/\mathcal{O}}(1))$ (which is finite, by the same argument showing boundedness of $h^0(F^+, W_m(1))$ in the proof of Lemma 2.23). We see that (2.25.2) will hold with $d_1 = h^0(F^+, W_{E/\mathcal{O}}(1))$ provided that the map (2.25.3) sends $H^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_m(1))$ into $H^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_N(1))$. Recalling the definition of our local conditions, this means we must show that if $v \in S_p$, then the map $H^1(F_v^+, W_m(1)) \to H^1(F_v^+, W_N(1))$ sends $\mathcal{L}^\perp_{v,m}$ to $\mathcal{L}^\perp_{v,N}$. By duality, we must show that if $v \in S_p$, then the map $H^1(F_v^+, W_N) \to H^1(F_v^+, W_m)$ induced by the surjection $W_N \to W_m$ sends $\mathcal{L}_{v,N}$ into $\mathcal{L}_{v,m}$. However, this follows immediately from the definitions. □

For Corollary 2.25 to be useful, we need to be able to find Taylor–Wiles data with good properties. To do this, we first introduce a useful definition.

Lemma 2.26. *Let $H \subset \mathrm{GL}_n(\mathcal{O})$ be a compact subgroup, and suppose the characteristic polynomial of every element of H splits over E (this condition is always satisfied after possibly enlarging E). Denote the associated representation of H by ρ, so we have $\mathcal{O}[H]$-modules $W = \mathrm{ad}\,\rho$, W_E, $W_{E/\mathcal{O}}$ as above. Then the following conditions are equivalent:*

1. *For all simple $E[H]$-submodules $V \subset W_E^0 = \mathrm{ad}^0\,\rho \otimes E$, we can find $h \in H$ with n distinct eigenvalues and $\alpha \in E$ such that α is an eigenvalue of h and $\mathrm{tr}\, e_{h,\alpha} V \neq 0$ (where $e_{h,\alpha} \in W_E$ denotes the h-equivariant projection to the α-eigenspace).*
2. *For all simple $E[H]$-submodules $V \subset W_E$, we can find $h \in H$ with n distinct eigenvalues and $\alpha \in E$ such that α is an eigenvalue of h and $\mathrm{tr}\, e_{h,\alpha} V \neq 0$.*
3. *For all non-zero $E[H]$-submodules $V \subset W_E$, there exists $h \in H$ with n distinct eigenvalues such that $V \not\subset (h-1)W_E$.*
4. *For all non-zero divisible $\mathcal{O}[H]$-submodules $V \subset W_{E/\mathcal{O}}$, there exists $h \in H$ with n distinct eigenvalues such that $V \not\subset (h-1)W_{E/\mathcal{O}}$.*

Proof. We note that (1) and (2) are equivalent because the scalar matrices $Z_E \subset W_E$ give a complement to W_E^0 in W_E, and the condition $\mathrm{tr}\, e_{h,\alpha} Z_E \neq 0$ is satisfied for any regular semisimple element $h \in H$ and eigenvalue $\alpha \in E$.

If $h \in \mathrm{GL}_n(\mathcal{O})$ has n distinct eigenvalues, then it acts semisimply on W_E. In particular, there is a unique h-invariant direct sum decomposition $W_E = W_E^h \oplus (h-1)W_E$. If $V \subset W_E$ is a h-invariant subspace, then there is a similar direct sum decomposition $V = V^h \oplus (h-1)V$. The condition that there exists an eigenvalue $\alpha \in E$ of h such that $\mathrm{tr}\, e_{h,\alpha} V \neq 0$ is equivalent to the condition that the projection of V to W_E^h is non-zero, or equivalently that $V^h \neq 0$. This is in turn equivalent to the condition that $V \not\subset (h-1)W_E$. This shows that (2) and (3) are equivalent.

Now we show that (3) and (4) are equivalent. For this we note that there is a $\mathrm{GL}_n(\mathcal{O})$-equivariant, inclusion-preserving bijection between the E-subspaces of W_E and the divisible $\mathcal{O}$-submodules of $W_{E/\mathcal{O}}$; this sends $V \subset W_E$ to $V + W/W$ and $V' \subset W_{E/\mathcal{O}}$ to

$$V = \{v \in W_E \mid \forall n \geq 0, \varpi^{-n} v \bmod W \in V'\}.$$

The proof in this case is complete on noting that $(h-1)W_E$ corresponds to $(h-1)W_{E/\mathcal{O}}$ under this bijection. □

Definition 2.27. *We say that a subgroup $H \subset \mathrm{GL}_n(\mathcal{O})$ is enormous if for all simple $E[H]$-submodules $V \subset W_E$, we can find $h \in H$ with n distinct eigenvalues in E and $\alpha \in E$ such that α is an eigenvalue of h and $\mathrm{tr}\, e_{h,\alpha} V \neq 0$.*

Remark 2.28. The above is a natural analogue of the definition of enormous subgroups in positive characteristic [KT17, Definition 4.10]. In contrast to the positive characteristic case, we do not need to assume any vanishing of cohomology groups for H. The necessary vanishing will follow from the purity of our Galois representations (see [Kis04, Lemma 6.2], this goes back to Serre for Tate modules of abelian varieties [Ser71]).

Lemma 2.29. *Let $H \subset \mathrm{GL}_n(\mathcal{O})$ be an enormous subgroup. Then H acts absolutely irreducibly on E^n, and in particular we have $H^0(H, W_E^0) = 0$.*

Proof. We need to show that the E-linear span of H in W_E is everything (by Burnside's theorem on matrix algebras). Consider

$$U = \{u \in W_E : \operatorname{tr}(hu) = 0 \quad \forall h \in H\}.$$

If U is non-zero, let V be a simple $E[H]$-submodule of U so we have an $h \in H$ with $\alpha \in E$ such that $\operatorname{tr} e_{h,\alpha} V \neq 0$. This is a contradiction, since $e_{h,\alpha}$ is a polynomial in h. We conclude that $U = 0$ and since the trace pairing on W_E is perfect the span of H is indeed W_E. (Compare with [Tho12, Appendix, Lemma 1].) □

Lemma 2.30. *Let $q \geq \operatorname{corank}_{\mathcal{O}} H^1(F_S/F^+, W_{E/\mathcal{O}}(1))$, and suppose that ρ satisfies the following conditions:*

1. *For all but finitely many finite places $v \nmid S$ of F, the eigenvalues of $\rho(\mathrm{Frob}_v)$ are algebraic numbers which have absolute value $q_v^{w/2}$ with respect to any complex embedding.*
2. *$\rho(G_{F(\zeta_{p^\infty})})$ is enormous.*

Then there exists $d \in \mathbf{N}$ such that for any $N \in \mathbf{N}$ we can find a Taylor–Wiles datum $(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \ldots, \alpha_{\widetilde{v},n})_{\widetilde{v} \in \widetilde{Q}})$ of level N with $|Q| = q$ such that

1. *for all $v \in Q$ and $i \neq j$ we have $\operatorname{ord}_\varpi(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j}) \leq d$*
2. *$h^1_{\mathcal{L}^\perp_{S \cup Q}}(F^+, W_N(1)) \leq d$.*

Proof. If Q is a set of Taylor–Wiles places then we have an exact sequence

$$0 \to H^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_N(1)) \to H^1_{\mathcal{L}^\perp_S}(F^+, W_N(1)) \to \bigoplus_{v \in Q} H^1(k(v), W_N(1)).$$

Suppose we could find $\sigma_1, \ldots, \sigma_q \in G_{F(\zeta_{p^\infty})}$ such that

(a) for each $i = 1, \ldots, q$, $\rho(\sigma_i)$ has n distinct eigenvalues in E;
(b) the kernel of the map

$$H^1(F_S/F^+, W_{E/\mathcal{O}}(1)) \to \bigoplus_{i=1}^q H^1(\widehat{\mathbf{Z}}, W_{E/\mathcal{O}}(1)) \cong \bigoplus_{i=1}^q W_{E/\mathcal{O}}(1)/(\sigma_i - 1)W_{E/\mathcal{O}}(1)$$

(product of restriction maps associated to the homomorphisms $\widehat{\mathbf{Z}} \to G_{F^+,S}$, the i^{th} such homomorphism sending 1 to σ_i) is a finite length $\mathcal{O}$-module.

Then consideration of the following diagram:

$$\begin{array}{ccccccccc}
0 \to & H^0(F^+, W_{E/\mathcal{O}}(1))/\varpi^N & \to & H^1(F_S/F^+, W_N(1)) & \to & H^1(F_S/F^+, W_{E/\mathcal{O}}(1))[\varpi^N] & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{i=1}^q H^0(\langle\sigma_i\rangle, W_{E/\mathcal{O}}(1))/\varpi^N & \to & \bigoplus_{i=1}^q H^1(\langle\sigma_i\rangle, W_N(1)) & \to & \bigoplus_{i=1}^q H^1(\langle\sigma_i\rangle, W_{E/\mathcal{O}}(1))[\varpi^N] & \to 0
\end{array}$$

shows that the kernel of the map

$$H^1(F_S/F^+, W_N(1)) \to \oplus_{i=1}^q H^1(\langle\sigma_i\rangle, W_N(1))$$

has length bounded independently of N (note that $H^0(F^+, W_{E/\mathcal{O}}(1))$ is a finite length $\mathcal{O}$-module). An application of the Chebotarev density theorem would then yield the theorem.

To complete the proof, it therefore suffices to show that for any non-zero homomorphism $f : E/\mathcal{O} \to H^1(F_S/F^+, W_{E/\mathcal{O}}(1))$, we can find $\sigma \in G_{F(\zeta_{p^\infty})}$ such that $\rho(\sigma)$ has n distinct eigenvalues in E and $\operatorname{Res}^{G_{F^+,S}}_{\langle\sigma\rangle} \circ f : E/\mathcal{O} \to W_{E/\mathcal{O}}(1)/(\sigma-1)W_{E/\mathcal{O}}(1)$

is still non-zero (as then we can argue by induction to get $\sigma_1, \dots, \sigma_q$ satisfying conditions (a), (b) above).

Let $F_\infty = F(\zeta_{p^\infty})$, let L'_∞/F^+ be the extension cut out by $W_E(1)$, and let $L_\infty = L'_\infty \cdot F_\infty$. Then $H^1(L'_\infty/F^+, W_E(1)) = 0$, by [Kis04, Lemma 6.2]. (It is in our appeal to this result that we make use of the purity assumption in the statement of the lemma.) The extension L_∞/L'_∞ is finite (as $E(\epsilon\delta_{F/F^+}) \subset W_E(1)$), so $H^1(L_\infty/F^+, W_E(1)) = 0$. It follows that $H^1(L_\infty/F^+, W_{E/\mathcal{O}}(1))$ is killed by a power of p and hence the homomorphism

$$\begin{aligned}\mathrm{Res}^{G_{F^+,S}}_{G_{L_\infty,S_{L_\infty}}} \circ f : E/\mathcal{O} \to & H^1(F_S/L_\infty, W_{E/\mathcal{O}}(1))^{G_{F^+,S}} \\ & \cong \mathrm{Hom}_{G_{F^+,S}}(G_{L_\infty,S_{L_\infty}}, W_{E/\mathcal{O}}(1))\end{aligned}$$

is still non-zero. Let $M \subset W_{E/\mathcal{O}}(1)$ be the $\mathcal{O}$-submodule generated by the elements $f(x)(\sigma)$, $x \in E/\mathcal{O}$, $\sigma \in G_{L_\infty}$; it is a non-zero divisible $\mathcal{O}[G_{F_\infty}]$-submodule of $W_{E/\mathcal{O}}(1)$. Using Lemma 2.26, we see that there exists $\sigma \in G_{F_\infty}$ such that $\rho(\sigma)$ has n distinct eigenvalues in E and $M \not\subset (\sigma - 1)W_{E/\mathcal{O}}(1)$. Consequently, there exists $m \geq 0$ and $\tau \in G_{L_\infty}$ such that $f(1/\varpi^m)(\tau) \notin (\sigma - 1)W_{E/\mathcal{O}}(1)$.

If $f(1/\varpi^m)(\sigma) \notin (\sigma - 1)W_{E/\mathcal{O}}(1)$, then we're done: $\mathrm{Res}^{G_{F^+,S}}_{\langle\sigma\rangle} \circ f$ is non-zero. Otherwise, we can assume $f(1/\varpi^m)(\sigma) \in (\sigma - 1)W_{E/\mathcal{O}}(1)$, and then $\mathrm{Res}^{G_{F^+,S}}_{\langle\tau\sigma\rangle} \circ f$ is non-zero. This completes the proof. □

Putting everything together, we obtain:

Corollary 2.31. *Let $q \geq \mathrm{corank}_{\mathcal{O}} H^1(F_S/F^+, W_{E/\mathcal{O}}(1))$, and suppose that ρ satisfies the following conditions:*

1. *For all but finitely many finite places $v \nmid S$ of F, the eigenvalues of $\rho(\mathrm{Frob}_v)$ are algebraic numbers which have absolute value $q_v^{w/2}$ with respect to any complex embedding.*
2. *For each $v \in S$, $\rho|_{G_{F_{\tilde{v}}}}$ is generic.*
3. *For each place $v|\infty$ of F^+, $\chi(c_v) = -1$.*
4. *$\rho(G_{F(\zeta_{p^\infty})})$ is enormous.*

Then there exists $d \in \mathbf{N}$ such that for each $N \in \mathbf{N}$ we can find a Taylor–Wiles datum Q_N of level N with $|Q_N| = q$ and a map

$$\mathcal{O}[\![x_1, \dots, x_{nq}]\!] \to R_{S \cup Q_N}$$

such that the images of the x_i are in $\mathfrak{q}_{S\cup Q_N}$ and

$$\mathfrak{q}_{S\cup Q_N}/(\mathfrak{q}^2_{S\cup Q_N}, x_1, \dots, x_{nq})$$

is a quotient of $(\mathcal{O}/\varpi^d)^{g_0}$, where $g_0 = g_0(S, \overline{\rho}, q)$ is as defined in the statement of Lemma 2.16.

Proof. Combining Proposition 2.20, Corollary 2.25 and Lemma 2.30 we deduce that there exists a constant d such that for each N we can find Q_N of level N and an $\mathcal{O}$-module map $\mathcal{O}^{nq} \to \mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N$ with cokernel killed by ϖ^d (note that the two $\mathcal{O}$-modules $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N$ and $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N}, \mathcal{O}/\varpi^N)$ are abstractly isomorphic). This allows us to define a map $\mathcal{O}[\![x_1, \dots, x_{nq}]\!] \to R_{S\cup Q_N}$ with the x_i mapping to images of generators of $\mathcal{O}^{nq}$ in $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N$, so that

$$\mathfrak{q}_{S\cup Q_N}/(\mathfrak{q}^2_{S\cup Q_N}, x_1, \dots, x_{nq}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N$$

is killed by ϖ^d. We need to explain how to deduce the slightly stronger result in the statement of the corollary. We first note that $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N}$ is a quotient of $\mathcal{O}^{g_0}$. Indeed, it is a finitely generated $\mathcal{O}$-module, and there is an isomorphism $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes_{\mathcal{O}} \mathcal{O}/\varpi \cong \mathfrak{m}_{R_{S\cup Q_N}}/(\mathfrak{m}^2_{R_{S\cup Q_N}}, \varpi)$, so we can apply Nakayama's lemma together with Lemma 2.16.

We may assume that $N > d$. In this case $M = \mathfrak{q}_{S\cup Q_N}/(\mathfrak{q}^2_{S\cup Q_N}, x_1, \ldots, x_{nq})$ is a quotient of $\mathcal{O}^{g_0}$ with the property that $M/(\varpi^N)$ is killed by ϖ^d. This is only possible if M is itself killed by ϖ^d, implying that M is a quotient of $(\mathcal{O}/\varpi^d)^{g_0}$. □

2.32. **Some examples of enormous subgroups.** Let $E/\mathbf{Q}_p$ be a coefficient field, and let $H \subset \mathrm{GL}_n(\mathcal{O})$ be a compact subgroup.

Lemma 2.33. *Suppose that for each $h \in H$, the characteristic polynomial of h has all of its roots in E.*

1. *Let $H' \subset H$ be a closed subgroup. If H' is enormous, then so is H.*
2. *Let $G \subset \mathrm{GL}_n$ denote the Zariski closure of H. If G° contains regular semisimple elements and acts absolutely irreducibly on E^n, then H is enormous.*

Proof. The first part is immediate from the definitions. For the second, we can assume that $G = G^\circ$. Since G acts absolutely irreducibly, $G(E)$ spans W_E. Let $H^{reg} \subset H$ denote the set of regular semisimple elements. It is Zariski dense in G. Indeed, by hypothesis G^{reg} is a non-empty Zariski open subset of G. The Zariski closure of H is contained in the union of the Zariski closure of H^{reg} and $G - G^{reg}$. This forces the Zariski closure of H^{reg} to be equal to G.

We must show that for any non-zero $v \in W_E$, there exists $h \in H^{reg}$ such that $\mathrm{tr}\, hv \neq 0$. If $\mathrm{tr}\, hv = 0$ for all $h \in H^{reg}$, then Zariski density implies that $\mathrm{tr}\, gv = 0$ for all $g \in G$. This contradicts the fact that the elements of $G(E)$ span W_E. □

Example 2.34. Let F be a totally real or CM number field, and let π be a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, and let $\rho : G_F \to \mathrm{GL}_n(\mathcal{O})$ be a model of $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ defined over $\mathcal{O}$. If $\mathrm{Sym}^2 \pi$ is cuspidal, then (after possibly enlarging E) $\rho(G_{F(\zeta_{p^\infty})})$ is an enormous subgroup of $\mathrm{GL}_n(\mathcal{O})$.

To see this, it is enough to note that the Zariski closure of the image of $r_{\pi,\iota}$ contains SL_2, and therefore that the Zariski closure of $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ also contains SL_2 (because passage to derived subgroup respects Zariski closure, cf. [Bor91, Ch. I, §2.1]). We can then appeal to Lemma 2.33.

We justify the claim that the Zariski closure of $r_{\pi,\iota}(G_F)$ contains SL_2. The identity component of the Zariski closure of $r_{\pi,\iota}(G_F)$ is a reductive subgroup of GL_2 which contains regular semisimple elements (by [Sen73, Theorem 1]). The only possibility we need to rule out is that $r_{\pi,\iota}(G_F)$ normalizes a maximal torus in GL_2. In this case, there is a quadratic extension F'/F such that $r_{\pi,\iota}|_{G_{F'}}$ is reducible. It's therefore enough to show that for any quadratic extension F'/F, $r_{\pi,\iota}|_{G_{F'}}$ is irreducible. We observe that if $r_{\pi,\iota}|_{G_{F'}}$ is reducible, then it's isomorphic to a sum $\chi_1 \oplus \chi_2$ of characters. Moreover, χ_1, χ_2 are de Rham and almost everywhere unramified, so therefore can be extended to compatible systems of 1-dimensional Galois representations. It follows that $r_{\pi,\iota'}|_{G_{F'}}$ is reducible for any other prime p' and isomorphism $\iota' : \overline{\mathbf{Q}}_{p'} \to \mathbf{C}$. In particular, $\mathrm{Sym}^2 r_{\pi,\iota'}$ is reducible. However, [BLGGT14, Theorem 5.5.2] implies that for a Dirichlet density 1 set of primes p', the representation $\mathrm{Sym}^2 r_{\pi,\iota'}$ is irreducible (note that for automorphic representations of $\mathrm{GL}_3(\mathbf{A}_F)$ the asssumption of 'extremely regular weight' in *loc. cit.* coincides with the usual notion of regular weight).

Example 2.35. Let F be a CM field, and let π be a polarizable automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ such that for some finite place v_0 of F, π_{v_0} is a twist of the Steinberg representation. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Then $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous.

Indeed, let G denote the Zariski closure of $r_{\pi,\iota}(G_F)$. Local-global compatibility at the place v_0 implies that G° contains a regular unipotent element (if $v_0|p$, we argue as in [KS19, Lemma 3.2]), so in particular it acts absolutely irreducibly on E^n. Then [Sch06, Proposition 4] (see also [Kat88, Classification Theorem 11.6]) shows that the derived group of G° is one of a finite list of possibilities, and that in any case it contains regular semisimple elements. We can again appeal to Lemma 2.33.

3. A result about Hecke algebras

Let p be a prime, let $n \geq 2$, and let $F_v/\mathbf{Q}_l$ be a finite extension for some $l \neq p$. Let $G = \mathrm{GL}_n(F_v)$, $U = \mathrm{GL}_n(\mathcal{O}_{F_v})$, and let $I \subset U$ be the standard Iwahori subgroup (i.e. the pre-image in U of the upper-triangular matrices in $GL_n(k(v))$). Let $E/\mathbf{Q}_p$ be a coefficient field. For $\mathcal{O}$ sufficiently large (i.e. containing a square root of q_v), the Iwahori Hecke algebra $\mathcal{H}_I = \mathcal{H}(G, I) \otimes_{\mathbf{Z}} \mathcal{O}$ has the Bernstein presentation

$$\mathcal{H}_I \cong \mathcal{O}[X_*(T)]\widetilde{\otimes}\mathcal{O}[I\backslash U/I].$$

The map $\mathcal{O}[X_*(T)] \to \mathcal{H}_I$ sends a dominant cocharacter $\lambda \in X_*(T)_+$ to the Hecke operator $q_v^{-l(\lambda)/2}[I\lambda(\varpi_v)I]$, where $l(\cdot)$ is the usual length function on the extended affine Weyl group. The twisted tensor product indicates the usual tensor product as $\mathcal{O}$-modules, with the algebra structure on $\mathcal{H}_I$ determined by the relations of [Lus89, Proposition 3.6]. We identify $\mathcal{O}[X_*(T)] = \mathcal{O}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ (x_i is the cocharacter embedding $\mathbf{G}_m$ into the ith diagonal entry of T). The centre, $Z(\mathcal{H}_I)$, is identified by the Bernstein presentation with the algebra of symmetric polynomials $\mathcal{O}[X_*(T)]^{S_n} = \mathcal{O}[e_1, e_2, \ldots, e_n, e_n^{-1}]$ ($e_1, \ldots, e_n$ are the usual elementary symmetric polynomials in $x_1, \ldots, x_n$).

Our identification of $\mathcal{O}[X_*(T)]$ with a polynomial algebra allows us to speak of polynomials as being elements of the Hecke algebra. In particular, we can think of $\Delta = \prod_{1 \leq i < j \leq n}(x_i - x_j)$ as being an element of $\mathcal{H}_I$, and its square Δ^2 as being an element of the centre $Z(\mathcal{H}_I)$.

To simplify notation, let $\mathcal{R} = \mathcal{O}[X_*(T)]^{S_n}$, $\mathcal{S} = \mathcal{O}[X_*(T)]$. Then $\mathcal{S}$ is a free $\mathcal{R}$-module, a basis being given by the monomials $x_{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ satisfying $0 \leq a_i \leq i - 1$ for each $i = 1, \ldots, n$. We write B for the set of tuples $\mathbf{a}$ satisfying these conditions.

If M is an $\mathcal{O}[\mathrm{GL}_n(F_v)]$-module M, then M^U is an $\mathcal{R}$-submodule of M^I (and in fact, if $z \in Z(\mathcal{H}_I)$ and $M \in M^U$, then we have the formula $z \cdot m = ([U]z) \cdot_U m$, where $\cdot_U$ denotes the action of $\mathcal{H}_U = \mathcal{H}(G, U) \otimes_{\mathbf{Z}} \mathcal{O}$ on M^U – see [HKP10, §4.6]). Thus there there is a canonical (and functorial) morphism

$$M^U \otimes_{\mathcal{R}} \mathcal{S} \to M^I, \tag{3.0.1}$$

given by the formula $m \otimes s \mapsto sm$. Since $\mathcal{S}$ is free over $\mathcal{R}$, $M^U \otimes_{\mathcal{R}} \mathcal{S}$ may be identified with $\oplus_{\mathbf{a} \in B} M^U$, and the above map with $(m_{\mathbf{a}})_{\mathbf{a} \in B} \mapsto \sum_{\mathbf{a} \in B} x_{\mathbf{a}} \cdot m_{\mathbf{a}}$.

The aim of this short section is to prove the following result, which will be applied in Section 4 (see Proposition 4.8).

Proposition 3.1. *Let $N \geq 1$, and let M be an $\mathcal{O}/\varpi^N[\mathrm{GL}_n(F_v)]$-module. Suppose that $q_v \equiv 1 \bmod \varpi^N$. Then the above morphism $M^U \otimes_{\mathcal{R}} \mathcal{S} \to M^I$ has kernel and cokernel annihilated by $\Delta^{n!}$.*

Note that $\Delta^{n!}$ always lies in $Z(\mathcal{H}_I)$. This is important for us since it means that in the global situation, $\Delta^{n!}$ will be in the image of the pseudodeformation ring (through which decomposition groups act via a homomorphism to the Bernstein centre).

Before proving the proposition, we establish an auxiliary result.

Lemma 3.2. *Consider the $n! \times n!$ matrix P with coefficients in $\mathbf{Z}[x_1, \dots, x_n]$ given by the formula $P_{\sigma,\mathbf{a}} = \sigma(x_{\mathbf{a}})$ ($\sigma \in S_n$, $\mathbf{a} \in B$). Then there exists a unique matrix $Q = (Q_{\mathbf{a},\sigma})$ with coefficients in $\mathbf{Z}[x_1, \dots, x_n]$ such that $PQ = QP = \Delta^{n!}$.*

Proof. It suffices to show existence, since then uniqueness follows by linear algebra over $\mathbf{Q}(x_1, \dots, x_n)$. The square of the determinant of P is equal to the discriminant of the ring extension $\mathbf{Z}[e_1, \dots, e_n] \to \mathbf{Z}[x_1, \dots, x_n]$. Using [Sta13, Tag 0C17], we see that the discriminant of this ring extension equals $\Delta^{n!}$ (the different ideal is generated by Δ). Therefore the determinant of P is equal to $\Delta^{n!/2}$, up to sign.

The existence of the adjugate matrix implies that there is a matrix Q' with coefficients in $\mathbf{Z}[x_1, \dots, x_n]$ such that $PQ' = \Delta^{n!/2}$. We then take $Q = \Delta^{n!/2}Q'$. $\square$

We observe that for all $\mathbf{a} \in B$, $\sigma, \tau \in S_n$, we have $\sigma(Q_{\mathbf{a},\tau}) = Q_{\mathbf{a},\sigma\tau}$. Indeed, this follows from the identity $\sigma(P)\sigma(Q) = \Delta^{n!}$ and the uniqueness of inverses.

Proof of Proposition 3.1. Since $q_v \equiv 1 \bmod \varpi^N$, we can identify $\mathcal{O}/\varpi^N[I\backslash U/I] = \mathcal{O}/\varpi^N[S_n]$, and $\mathcal{H}(G, I) \otimes_{\mathbf{Z}} \mathcal{O}/\varpi^N$ with the group algebra of the extended affine Weyl group $X_*(T) \rtimes S_n$ (because the Iwahori–Matsumoto relations are a q-deformation of the relations defining the group algebra of the affine Weyl group). Let $e = \sum_{\sigma \in S_n} \sigma \in \mathcal{O}/\varpi^N[S_n] \subset \mathcal{H}(G, I) \otimes_{\mathbf{Z}} \mathcal{O}/\varpi^N$. Then $e = [U]$, so in particular $eM^I \subset M^U$. Recalling that $[I]$ is the unit of $\mathcal{H}_I$, we note that e need not be an idempotent, since $e^2 = [U : I]e$ (note that $q_v \equiv 1 \bmod \varpi^N \implies [U : I] \equiv n! \bmod \varpi^N$, and we do not rule out the case $p \leq n$).

We have defined a map $f : \oplus_{\mathbf{a} \in B} M^U \to M^I$ by the formula $(m_{\mathbf{a}})_{\mathbf{a} \in B} \mapsto \sum_{\mathbf{a} \in B} x_{\mathbf{a}} \cdot m_{\mathbf{a}}$. We define a map $g : M^I \to \oplus_{\mathbf{a} \in B} M^U$ by the formula $g(m) = (eQ_{\mathbf{a},1}m)_{\mathbf{a} \in B}$.

We now compute $f \circ g$ and $g \circ f$. We have for $m \in M^I$

$$f(g(m)) = \sum_{\mathbf{a} \in B} x_{\mathbf{a}} e Q_{\mathbf{a},1} m = \sum_{\mathbf{a} \in B} \sum_{\sigma \in S_n} x_{\mathbf{a}} \sigma(Q_{\mathbf{a},1}) \sigma(m).$$

This in turn we can rewrite as

$$\sum_{\sigma \in S_n} \sum_{\mathbf{a} \in B} P_{1,\mathbf{a}} Q_{\mathbf{a},\sigma} \sigma(m) = \Delta^{n!} m.$$

Similarly, we have for $m = (m_{\mathbf{a}})_{\mathbf{a} \in B} \in \oplus_{\mathbf{a} \in B} M^U$:

$$g(f(m))_{\mathbf{a}} = eQ_{\mathbf{a},1} \sum_{\mathbf{b} \in B} x_{\mathbf{b}} \cdot m_{\mathbf{b}} = \sum_{\sigma \in S_n} \sum_{\mathbf{b} \in B} Q_{\mathbf{a},\sigma} P_{\sigma,\mathbf{b}} \sigma(m_{\mathbf{b}}).$$

Note that S_n acts trivially on M^U. We can therefore rewrite the above expression as

$$\sum_{\mathbf{b}\in B}\sum_{\sigma\in S_n} Q_{\mathbf{a},\sigma}P_{\sigma,\mathbf{b}}m_{\mathbf{b}} = \Delta^{n!}m_{\mathbf{a}}.$$

This completes the proof. □

4. Patching

In this section we prove our main technical result (Theorem 4.2).

4.1. **Set-up.** We suppose given the following data:

- A CM number field F with maximal totally real subfield F^+. We assume that F/F^+ is everywhere unramified and that $[F^+:\mathbf{Q}]$ is even.
- An integer $n \geq 2$ and a cuspidal, polarized, regular algebraic automorphic representation $(\pi, \delta^n_{F/F^+})$ of $\mathrm{GL}_n(\mathbf{A}_F)$ (i.e. π is of unitary type).
- A prime p and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. We assume that for each place $w|p$ of F, π_w has an Iwahori-fixed vector.
- A finite set S of finite places of F^+, containing the set S_p of p-adic places and all places above which π is ramified. We assume that each place of S splits in F.

We recall that under these conditions we define an extension of $r_{\pi,\iota}$ to a homomorphism to $\mathcal{G}_n$, which then gives the action of G_{F^+} on $\mathrm{ad}\, r_{\pi,\iota}$ (see §1).

Theorem 4.2. *With set-up as above, assume moreover that $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous. Then*

$$H^1_f(F^+, \mathrm{ad}\, r_{\pi,\iota}) = 0.$$

We note here that the assumptions of Lemma 2.23 hold for $r_{\pi,\iota}$ by [BLGGT14, Theorem 2.1.1] (which collects together results of many people).

The proof of Theorem 4.2 will use automorphic forms on definite unitary groups. To this end, we can find the following data:

- For each place $v \in S$, a choice of place $\widetilde{v}$ of F lying above v. We set $\widetilde{S} = \{\widetilde{v} \mid v \in S\}$ and $\widetilde{S}_p = \{\widetilde{v} \mid v \in S_p\}$.
- A Hermitian form $\langle \cdot, \cdot \rangle : F^n \times F^n \to F$ such that the associated unitary group G (defined on R-points by $G(R) = \{g \in \mathrm{GL}_n(F \otimes_{F^+} R) \mid g^* g = 1\}$) is definite at infinity and quasi-split at each finite place of F^+.
- A reductive group scheme over $\mathcal{O}_{F^+}$ extending G.
- For each finite place $v = ww^c$ of F^+ which splits in F, an isomorphism $\iota_w : G_{\mathcal{O}_{F_v^+}} \to \mathrm{Res}_{\mathcal{O}_{F_w}/\mathcal{O}_{F_v^+}} \mathrm{GL}_n$ of group schemes over $\mathcal{O}_{F_v^+}$. We assume that the induced isomorphism $\iota_w : G(F_v^+) \to \mathrm{GL}_n(F_w)$ is in the same inner class as the isomorphism given by inclusion $G(F_v^+) \subset \mathrm{GL}_n(F_w) \times \mathrm{GL}_n(F_{w^c})$, followed by projection to the first factor.
- An automorphic representation σ of $G(\mathbf{A}_{F^+})$ with the following properties:
  - For each finite inert place v of F^+, $\sigma_v^{G(\mathcal{O}_{F_v^+})} \neq 0$ and σ_v, π_v are related by unramified base change.
  - For each split place $v = ww^c$ of F^+, $\sigma_v \cong \pi_w \circ \iota_w$.
  - If $v|\infty$ is a place of F^+, then the infinitesimal character of σ_v respects that of π_v under base change. (We recall this relation more precisely below.)

- An open compact subgroup $U = \prod_v U_v$ of $G(\mathbf{A}_{F^+}^\infty)$ with the following properties:
  - For each place $v \in S_p$, $U_v = \iota_{\widetilde{v}}^{-1}(\mathrm{Iw}_{\widetilde{v}})$, where $\mathrm{Iw}_{\widetilde{v}} \subset \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ is the standard Iwahori subgroup (defined as in §3).
  - For each inert place v of F^+, $U_v = G(\mathcal{O}_{F_v^+})$.
  - $(\sigma^\infty)^U \neq 0$.
  - U is sufficiently small: for all $g \in G(\mathbf{A}_{F^+}^\infty)$, $gUg^{-1} \cap G(F^+) = \{1\}$.

(We can find such a G because $[F^+ : \mathbf{Q}]$ is even. The existence of σ is deduced from that of π using [Lab11, §5].) We can regard σ_∞ as an algebraic representation of the group $(\mathrm{Res}_{F^+/\mathbf{Q}}\, G)_{\mathbf{C}}$. Let $\widetilde{I}_p \subset \mathrm{Hom}(F, \overline{\mathbf{Q}}_p)$ denote the set of embeddings inducing places $\widetilde{v} \in \widetilde{S}_p$. Then our choices determine an isomorphism

$$(\mathrm{Res}_{F^+/\mathbf{Q}}\, G)_{\overline{\mathbf{Q}}_p} \cong \prod_{\tau \in \widetilde{I}_p} \mathrm{GL}_n.$$

Let $\lambda = (\lambda_\tau)_{\tau \in \widetilde{I}_p} \in (\mathbf{Z}_+^n)^{\widetilde{I}_p}$ denote the highest weight of the algebraic representation V_λ of $(\mathrm{Res}_{F^+/\mathbf{Q}}\, G)_{\overline{\mathbf{Q}}_p}$ such that $V_\lambda \otimes_{\iota, \overline{\mathbf{Q}}_p} \mathbf{C} \cong \sigma_\infty^\vee$. We can define a highest weight ξ for $(\mathrm{Res}_{F/\mathbf{Q}}\, \mathrm{GL}_n)_{\overline{\mathbf{Q}}_p}$ by letting $\xi_\tau = \lambda_\tau$ and $\xi_{\tau c} = -w_0 \lambda_\tau$ for $\tau \in \widetilde{I}_p$ (w_0 is the longest element in the Weyl group of GL_n). The infinitesimal character of π_∞ is the same as that of $V_\xi^\vee \otimes_{\iota, \overline{\mathbf{Q}}_p} \mathbf{C}$.

The Hodge–Tate weights of $r_{\pi,\iota}$ may be described as follows: if $\tau \in \widetilde{I}_p$, then

$$\mathrm{HT}_\tau(r_{\pi,\iota}) = \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \dots, \lambda_{\tau,n}\}.$$

We fix once and for all integers $a \leq b$ such that for all $\tau \in \mathrm{Hom}(F, \overline{\mathbf{Q}}_p)$, the elements of $\mathrm{HT}_\tau(r_{\pi,\iota})$ are contained in $[a, b]$ and $a + b = n - 1$.

Let $E/\mathbf{Q}_p$ be a coefficient field containing the image of every embedding $F \to \overline{\mathbf{Q}}_p$. After possibly enlarging E, we can assume that there is a model $\rho : G_{F,S} \to \mathrm{GL}_n(\mathcal{O})$ of $r_{\pi,\iota}$, which extends to a homomorphism $r : G_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$ such that $\nu \circ r = \epsilon^{1-n}\delta_{F/F^+}^n$. Let $\overline{D}$ denote the group determinant of $\overline{\rho}$, which is then defined over k.

With these choices the pseudodeformation ring denoted R_S in §2.19 is defined, as well as the prime ideal $\mathfrak{q}_S = \ker(R_S \to \mathcal{O})$ determined by ρ. Moreover, for any Taylor–Wiles datum $(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \dots, \alpha_{\widetilde{v},n})_{v \in Q})$ we have the auxiliary ring $R_{S \cup Q}$. We introduce one more object here: it is the maximal quotient $R_{S \cup Q} \to R_{S \cup Q, ab}$ over which for each $v \in Q$, the restriction of the universal pseudocharacter to $W_{F_{\widetilde{v}}}$ factors through $W_{F_{\widetilde{v}}}^{ab}$. Thus we have a diagram

$$R_{S \cup Q} \to R_{S \cup Q, ab} \to R_S.$$

4.3. **Hecke algebras.** We can find a representation $\mathcal{V}_\lambda$ of the group scheme $(\mathrm{Res}_{\mathcal{O}_{F^+}/\mathbf{Z}}\, G)_{\mathcal{O}}$, finite free over $\mathcal{O}$, and such that $\mathcal{V}_\lambda \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p \cong V_\lambda$. (For example, use the construction of [Ger19, §2.2].) Thus $\mathcal{V}_\lambda(\mathcal{O})$ is a finite free $\mathcal{O}$-module which receives an action of $U_p = \prod_{v \in S_p} U_v$. For any open compact subgroup $V = \prod_v V_v \subset U$, and any $\mathcal{O}$-algebra A, we define $S_\lambda(V, A)$ to be the set of functions $f : G(\mathbf{A}_{F^+}^\infty) \to \mathcal{V}_\lambda(A)$ such that for each $v \in V$, $\gamma \in G(F^+)$, $g \in G(\mathbf{A}_{F^+}^\infty)$, $v_p f(\gamma g v) = f(g)$. We observe that

$$\varinjlim_{U^p} S_\lambda(U^p U_p, A)$$

has a natural structure of $A[U^p]$-module, and the U^p-invariants are $S_\lambda(U, A)$. It follows that $S_\lambda(U, A)$ has a natural structure of $\mathcal{H}(G(\mathbf{A}_{F^+}^{\infty;p}), U^p)$-module. A standard argument (cf. [Ger19, Lemma 2.2.5]) shows that there is an isomorphism of $\mathcal{H}(G(\mathbf{A}_{F^+}^{\infty;p}), U^p)$-modules

$$S_\lambda(U, \mathcal{O}) \otimes_{\iota,\mathcal{O}} \mathbf{C} \cong \oplus_\mu (\mu^\infty)^U,$$

where the sum is over automorphic representations of $G(\mathbf{A}_{F^+})$ (with multiplicity) such that $\mu_\infty \cong \sigma_\infty$.

If $V = \prod_v V_v$ is an open compact subgroup of U and T is a finite set of places of F^+ containing all places such that $V_v \neq G(\mathcal{O}_{F_v^+})$, then we write $\mathbf{T}_\lambda^T(V, A)$ for the A-subalgebra of $\mathrm{End}_A(S_\lambda(V, A))$ generated by the unramified Hecke operators at split places away from T. After possibly enlarging E, the existence of σ implies the existence of a homomorphism

$$h_{V,\sigma} : \mathbf{T}_\lambda^T(V, \mathcal{O}) \to \mathcal{O}$$

giving the Hecke eigenvalues of $\iota^{-1}\sigma^\infty$. On the other hand, the results of [Lab11, §5] imply the existence of a group determinant $D_{V,\lambda}$ of $G_{F,T}$ valued in $\mathbf{T}_\lambda^T(V, \mathcal{O})$ (construction as in [Tho15, Proposition 4.11]).

Let $\mathfrak{m} \subset \mathbf{T}_\lambda^S(U, \mathcal{O})$ denote the unique maximal ideal containing $\ker h_{U,\sigma}$, and set

$$S_\emptyset = S_\lambda(U, \mathcal{O})_\mathfrak{m}, \mathbf{T}_\emptyset = \mathbf{T}_\lambda^S(U, \mathcal{O})_\mathfrak{m}.$$

Then there is a surjective homomorphism $R_{\overline{D},S} \to \mathbf{T}_\emptyset$ classifying $D_{U,\lambda}$.

Lemma 4.4. *The map $R_{\overline{D},S} \to \mathbf{T}_\emptyset$ factors through the quotient R_S.*

Proof. If we invert p then $\mathbf{T}_\emptyset \otimes_\mathcal{O} \overline{\mathbf{Q}}_p = \prod_\mu E_\mu$ is a product of fields indexed by automorphic representations μ of $G(\mathbf{A}_{F^+})$ with $\mu_\mathfrak{m}^U \neq 0$ and $\mu_\infty \cong \sigma_\infty$. To prove the lemma, it suffices to show that each of the maps $R_{\overline{D},S} \to E_\mu$ factors through the quotient R_S: in other words, the conjugate self-duality condition and the semi-stability condition of (2.16.1). These conditions follow from local–global compatibility for the Galois representation associated to the base change of μ. □

4.5. **Automorphic data associated to Taylor–Wiles data.** Suppose given a set Q of Taylor–Wiles places. In this case we define open compact subgroups $U_0(Q) = \prod_v U_0(Q)_v$ and $U_1(Q) = \prod_v U_1(Q)_v$ as follows:

- If $v \notin Q$, then $U_0(Q)_v = U_1(Q)_v = U_v$.
- If $v \in Q$, then $U_0(Q)_v = \iota_{\widetilde{v}}^{-1}(\mathrm{Iw}_{\widetilde{v}})$ and $U_1(Q)_v$ is the smallest open subgroup of $U_1(Q)_v$ such that $U_0(Q)_v/U_1(Q)_v$ is a p-group.

We set $\Delta_Q = U_0(Q)/U_1(Q)$, which may be naturally identified with $\prod_{v\in Q} k(v)^\times(p)^n$. We write $\mathfrak{m}_Q$ for the intersection of $\mathfrak{m}$ with $\mathbf{T}_\lambda^{S\cup Q}(U, \mathcal{O})$, $\mathfrak{m}_{0,Q}$ for the pre-image of $\mathfrak{m}_Q$ in $\mathbf{T}_\lambda^{S\cup Q}(U_0(Q), \mathcal{O})$, and $\mathfrak{m}_{1,Q}$ for the pre-image of $\mathfrak{m}_{0,Q}$ in $\mathbf{T}_\lambda^{S\cup Q}(U_1(Q), \mathcal{O})$. We define

$$\mathbf{T}_{0,Q} = \mathbf{T}_\lambda^{S\cup Q}(U_0(Q), \mathcal{O})_{\mathfrak{m}_{0,Q}}, \mathbf{T}_Q = \mathbf{T}_\lambda^{S\cup Q}(U_1(Q), \mathcal{O})_{\mathfrak{m}_{1,Q}}.$$

As in Lemma 4.4, we have a surjective map $R_{S\cup Q} \to \mathbf{T}_Q$. Note that the natural map $\mathbf{T}_\lambda^{S\cup Q}(U, \mathcal{O})_{\mathfrak{m}_Q} \to \mathbf{T}_\emptyset$ is in fact an isomorphism, and so there are surjections

$$\mathbf{T}_Q \to \mathbf{T}_{0,Q} \to \mathbf{T}_\emptyset.$$

So far we have not used any Hecke operators at places $v \in Q$. For any $v \in Q$, $\alpha \in F_{\widetilde{v}}^\times$, and $1 \leq i \leq n$, we let $t_{v,i} : F_{\widetilde{v}}^\times \to \mathcal{H}(G(F_v^+), U_1(Q)_v)$ denote the composite with $\iota_{\widetilde{v}}^{-1}$ of the homomorphism defined just above [ACC+18, Proposition 2.2.7] (and denoted $t_{v,i}$ there). That proposition shows that if π_v is an irreducible admissible $\overline{\mathbf{Q}}_p[G(F_v^+)]$-module such that $\pi_v^{U_1(Q)_v} \neq 0$, then for any $\sigma \in W_{F_{\widetilde{v}}}$ and $\alpha \in F_{\widetilde{v}}^\times$ such that $\mathrm{Art}_{F_{\widetilde{v}}}(\alpha) = \sigma|_{F_{\widetilde{v}}^{ab}}$, the characteristic polynomial of $\mathrm{rec}^T_{F_{\widetilde{v}}}(\pi_v \circ \iota_{\widetilde{v}}^{-1})$ on σ equals

$$\sum_{i=0}^{n} (-1)^i X^{n-i} e_{v,i}(\alpha, \pi_v), \tag{4.5.1}$$

where $e_{v,i}(\alpha) \in \mathcal{H}(G(F_v^+), U_1(Q)_v)$ is the ith elementary symmetric polynomial in $t_{v,1}(\alpha), \dots, t_{v,n}(\alpha)$, and $e_{v,i}(\alpha, \pi_v) \in \overline{\mathbf{Q}}_p$ is the scalar by which it acts on $\pi_v^{U_1(Q)_v}$. The elements $e_{v,i}(\alpha)$ generate the centre of $\mathcal{H}(G(F_v^+), U_1(Q)_v) \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p$, by [Fli11, Proposition 4.11].

We define $\mathbf{T}_{0,Q}^Q \subset \mathrm{End}(S_\lambda(U_0(Q), \mathcal{O})_{\mathfrak{m}_{0,Q}})$ to be the subalgebra generated by $\mathbf{T}_{0,Q}$ and the elements $t_{v,i}(\alpha)$ for all $v \in Q$, $i = 1, \dots, n$ and $\alpha \in F_{\widetilde{v}}^\times$. We define $\mathbf{T}_Q^Q \subset \mathrm{End}(S_\lambda(U_1(Q), \mathcal{O})_{\mathfrak{m}_{1,Q}})$ similarly. Thus $\mathbf{T}_Q^Q$ is an $\mathcal{O}[\Delta_Q]$-algebra (the image of $\mathcal{O}[\Delta_Q]$ in $\mathbf{T}_Q^Q$ is generated by the elements $t_{v,i}(\alpha)$ with $\alpha \in \mathcal{O}_{F_{\widetilde{v}}}^\times$). Neither $\mathbf{T}_{0,Q}^Q$ nor $\mathbf{T}_Q^Q$ need be local rings. We denote by $\mathbf{a}_Q$ the augmentation ideal of $\mathcal{O}[\Delta_Q]$.

Lemma 4.6. *$S_\lambda(U_1(Q), \mathcal{O})$ is a free $\mathcal{O}[\Delta_Q]$-module and the trace map induces $S_\lambda(U_1(Q), \mathcal{O})/\mathbf{a}_Q \cong S_\lambda(U_0(Q), \mathcal{O})$.*

Proof. The proof is identical to that of [CHT08, Lemma 3.3.1], using that U (and hence any subgroup of U) is sufficiently small. □

We let $A_Q = \otimes_{v \in Q} \mathcal{O}[t_{v,1}(\varpi_v)^{\pm 1}, \dots, t_{v,n}(\varpi_v)^{\pm 1}]$. This is a polynomial subalgebra of $\otimes_{v \in Q} \mathcal{H}(G(F_v^+), U_0(Q)_v)$ that receives an action of the group $W_Q = \prod_{v \in Q} S_n$. For every $m \geq 1$, we have a canonical morphism of $\mathbf{T}_Q$-modules

$$\eta_{Q,m} : S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}} \otimes_{A_Q^{W_Q}} A_Q \to S_\lambda(U_0(Q), \mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q}},$$

as in (3.0.1).

For each $v \in Q$, the universal pseudocharacter over $R_{S \cup Q, ab}$ determines by restriction an n-dimensional pseudocharacter γ_v of $W_{F_{\widetilde{v}}}^{ab}$ valued in $R_{S \cup Q, ab}$. Each restriction $\gamma_v|_{I_{F_{\widetilde{v}}}}$ factors through the quotient $k(v)^\times(p)$ of $\mathrm{Art}_{F_{\widetilde{v}}}(\mathcal{O}_{F_{\widetilde{v}}}^\times)$ (compare [Che14, Lemma 3.8]).

On the other hand, for each $i = 1, \dots, n$, there is a character $\alpha_{v,i} : W_{F_{\widetilde{v}}}^{ab} \to (\mathbf{T}_Q^Q)^\times$ given by the formula $\alpha_{v,i}(\mathrm{Art}_{F_{\widetilde{v}}}(\alpha)) = t_{v,i}(\alpha)$. We write α_v for the pseudocharacter $\alpha_v = \alpha_{v,1} \oplus \dots \oplus \alpha_{v,n}$.

These two families of pseudocharacters are related by the following lemma, which is a formulation of local–global compatibility at $v \in Q$.

Lemma 4.7. *(1) The map*

$$R_{S \cup Q} \to \mathbf{T}_Q$$

factors through the quotient $R_{S \cup Q, ab}$.

(2) Let $v \in Q$. The composite of γ_v with the map $R_{S \cup Q, ab} \to \mathbf{T}_Q \hookrightarrow \mathbf{T}_Q^Q$ equals α_v.

(3) The image of the map

$$R_{S \cup Q, ab} \to \mathbf{T}_Q \hookrightarrow \mathbf{T}_Q^Q$$

contains the Hecke operators $e_{v,i}(\alpha)$ *for each* $v \in Q, i = 1, \cdots, n$ *and* $\alpha \in F_{\widetilde{v}}^{\times}$.

Proof. If we invert p then $\mathbf{T}_Q \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p = \prod_\mu E_\mu$ is a product of fields indexed by automorphic representations μ of $G(\mathbf{A}_{F^+})$ with $\mu_{\mathfrak{m}_{1,Q}}^{U_1(Q)} \neq 0$ and $\mu_\infty \cong \sigma_\infty$. To prove the first part of the lemma, it suffices to show that each of the maps $R_{S \cup Q} \to E_\mu$ factors through the quotient $R_{S\cup Q,ab}$. This follows from [ACC+18, Prop. 2.2.7]. The second and third parts of the lemma follow from the formula (4.5.1) which computes the characteristic polynomials of $\mathrm{rec}^T_{F_{\widetilde{v}}}(\mu_{\widetilde{v}} \circ \iota_{\widetilde{v}}^{-1})$ evaluated on elements of $W_{F_{\widetilde{v}}}$. □

We caution the reader that the map $R_{S\cup Q,ab} \to \mathbf{T}_Q^Q$ is not in general surjective, because of the presence of Hecke operators at Q which do not lie in the Bernstein centre.

The following proposition will be crucial for controlling our patched modules of automorphic forms. As mentioned in the introduction, this is inspired by arguments of Pan [Pan19].

Proposition 4.8. *Fix* $d \in \mathbf{N}$. *There exists a constant* $c \in \mathbf{N}$ *(depending only on* d*) such that, for any* N *and any Taylor–Wiles datum* $(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \ldots, \alpha_{\widetilde{v},n})_{\widetilde{v}\in\widetilde{Q}})$ *for* $r_{\pi,\iota}$ *of level* N *satisfying*

$$\sum_{v\in Q} \sum_{1\le i<j\le n} \mathrm{ord}_\varpi(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j}) \le d,$$

there is an element $f_Q \in R_{S\cup Q,ab}$ *such that*

1. f_Q *kills the kernel and cokernel of* $\eta_{Q,m}$ *for all* $m \le N$
2. *The image* $f_{Q,\sigma}$ *of* f_Q *under the composition of maps*
$$R_{S\cup Q,ab} \to \mathbf{T}_Q \xrightarrow{h_{U_1(Q),\sigma}} \mathcal{O}$$
satisfies $\mathrm{ord}_\varpi(f_{Q,\sigma}) \le c$.

Proof. We set

$$\overline{f}_Q = \left(\prod_{v\in Q} \prod_{1\le i<j\le n} (t_{v,i}(\varpi_v) - t_{v,j}(\varpi_v))^{n!} \right) \in \mathbf{T}_Q^Q$$

and let f_Q be a pre-image of $\overline{f}_Q$ in $R_{S\cup Q,ab}$ (such a pre-image exists by Lemma 4.7). It follows from Proposition 3.1 that f_Q kills the kernel and cokernel of $\eta_{Q,m}$ for all $m \le N$. If we take $c = n!d$ then, again using Lemma 4.7, we see that the second part of the proposition is satisfied. □

We give one last piece of structure. Suppose fixed an ordering $Q = \{v_1, \ldots, v_q\}$ and for each $v \in Q$ a surjection $\mathbf{Z}_p \to k(v)^\times(p)$. This data determines a surjection $(\mathbf{Z}_p^n)^q \to \prod_{v\in Q} k(v)^\times(p)^n = \Delta_Q$, hence a surjective algebra homomorphism $S_\infty \to \mathcal{O}[\Delta_Q]$, where $S_\infty = \mathcal{O}[\![y_1^{(i)}, \ldots, y_q^{(i)} : 1 \le i \le n]\!]$. The group $W_Q = \prod_{v\in Q} S_n$ acts on S_∞ by permutation of co-ordinates, and the invariant subring $S_\infty^{W_Q}$ may be identified with $\mathcal{O}[\![e_1^{(i)}, \ldots, e_q^{(i)} : 1 \le i \le n]\!]$, where $e_j^{(i)}$ is the ith elementary symmetric polynomial in $y_j^{(1)}, \ldots, y_j^{(n)}$. The ring $S_\infty^{W_Q}$ also has a role to play as a consequence of the following easy lemma:

Lemma 4.9. *The functor of deformations of the trivial pseudocharacter of* $\mathbf{Z}_p$ *of dimension* n *is represented by* $\mathcal{O}[\![X_1, \dots, X_n]\!]^{\mathfrak{S}_n}$*, with the universal characteristic polynomial* $\chi(t)$ *of* $1 \in \mathbf{Z}_p$ *given by* $\prod_{i=1}^n ((t-1) - X_i)$.

Proof. Indeed, a residually trivial pseudocharacter of $\mathbf{Z}_p$ of dimension n over a ring $A \in \mathcal{C}_{\mathcal{O}}$ is precisely a point of $(\mathrm{GL}_n /\!\!/ \mathrm{GL}_n)(A)$ lying over the image of the identity in $(\mathrm{GL}_n /\!\!/ \mathrm{GL}_n)(k)$. Now we use the identification of the adjoint quotient $\mathrm{GL}_n /\!\!/ \mathrm{GL}_n$ with the quotient of the diagonal maximal torus by the Weyl group. The universal deformation is given by the orbit of the matrix $\mathrm{diag}(1 + X_1, \dots, 1 + X_n)$. □

Consequently, there is a homomorphism $S_\infty^{W_Q} \to R_{S \cup Q, ab}$, classifying the pullback of the tuple $(\gamma_v)_{v \in Q}$ to a tuple of n-dimensional pseudocharacters of the group $\mathbf{Z}_p$. There is also a homomorphism $S_\infty^{W_Q} \to \mathbf{T}_Q^Q$, classifying the pullback of $(\alpha_v)_{v \in Q}$ to a tuple of n-dimensional pseudocharacters of the group $\mathbf{Z}_p$. This coincides with the restriction to $S_\infty^{W_Q}$ of the homomorphism $S_\infty \to \mathbf{T}_Q^Q$ determined by the $\mathcal{O}[\Delta_Q]$-algebra structure on $\mathbf{T}_Q^Q$. Lemma 4.7 has the following corollary.

Lemma 4.10. *The map* $S_\infty^{W_Q} \to \mathbf{T}_Q^Q$ *factors through* $\mathbf{T}_Q$*, and the map* $R_{S \cup Q, ab} \to \mathbf{T}_Q$ *is a homomorphism of* $S_\infty^{W_Q}$*-algebras.*

4.11. **The patching argument.**

- Fix $q = \mathrm{corank}_{\mathcal{O}} H^1(F_S/F^+, \mathrm{ad}\,\rho(1) \otimes_{\mathcal{O}} E/\mathcal{O})$. Applying Corollary 2.31, we fix for each $N \geq 1$ a Taylor–Wiles datum Q_N of level N, and we write $\Delta_N = \Delta_{Q_N}$, $\mathbf{a}_N = \mathbf{a}_{Q_N}$, $R_N = R_{S \cup Q_N, ab}$, $\mathbf{T}_N = \mathbf{T}_{Q_N}$. We set $R_0 = R_S$. We set $\mathfrak{q}_N = \ker(R_N \xrightarrow{h_{U_1(Q_N),\sigma}} \mathcal{O})$ and $\mathfrak{q}_0 = \ker(R_0 \xrightarrow{h_{U,\sigma}} \mathcal{O})$. Thus $\mathfrak{q}_N$ is the pre-image of $\mathfrak{q}_0$ under the natural map $R_N \to R_0$.
- Let $S_\infty = \mathcal{O}[\![y_1^{(i)}, \dots, y_q^{(i)} : 1 \leq i \leq n]\!]$ and fix orderings of Q_N and generators of $k(v)^\times(p)$ for all N and all $v \in Q_N$ and thus surjective maps $S_\infty \to \mathcal{O}[\Delta_N]$. Let $\mathbf{a}_\infty \subset S_\infty$ be the augmentation ideal (equal to the inverse image of $\mathbf{a}_N$ under each of the previously defined maps).
- We moreover fix uniformisers ϖ_v for all $v \in Q_N$ (for every N). This allows us to think of the pseudocharacters γ_v as pseudocharacters of $k(v)^\times(p) \times \mathbf{Z}$. Recalling that we have fixed a generator of $k(v)^\times(p)$ and an ordering on Q_N, for every N we have a q-tuple $(\gamma_{N,1}, \dots, \gamma_{N,q})$ of n-dimensional pseudocharacters of $(\mathbf{Z}_p \times \mathbf{Z})$ with coefficients in R_N.
- We have actions of $t_{v,i}(\varpi_v)$ on $S_\lambda(U_0(Q_N), \mathcal{O})_{\mathfrak{m}_{0,Q}}$ and $S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q}}$ for each $v \in Q_N$ and $i = 1, \dots, n$. Using these actions, together with the fixed orderings on Q_N, we obtain an action of the algebra
$$A = \otimes_{j=1}^q \mathcal{O}[(t_j^{(1)})^{\pm 1}, \dots, (t_j^{(n)})^{\pm 1}]$$
on these spaces, together with an identification of A with A_{Q_N} sending $t_j^{(i)}$ to $t_{v_j,i}(\varpi_{v_j})$. We have characters $\alpha_j^{(i)} : \mathbf{Z}_p \times \mathbf{Z} \to (S_\infty \otimes_{\mathcal{O}} A)^\times$ for $i = 1, \dots, n$ and $j = 1, \dots, q$. By Lemma 4.7, the pushforward of the pseudocharacter $\alpha_j = \mathrm{tr}\,\alpha_j^{(1)} \oplus \cdots \oplus \alpha_j^{(n)}$ to $\mathrm{End}_{\mathcal{O}}(S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q}})$ takes values in $\mathbf{T}_N$ and equals the pushforward of $\gamma_{N,j}$ there.
- We can identify all the Weyl groups W_{Q_N} (using our fixed orderings of Q_N for each N). We denote them all by W. There is a natural W-action on S_∞, compatible with the maps to $\mathcal{O}[\Delta_N]$. The invariants S_∞^W are a

regular local $\mathcal{O}$-algebra, with S_∞ a finite free S_∞^W-algebra (S_∞^W is a power series algebra over the elementary symmetric polynomials in $y_j^{(1)}, \ldots, y_j^{(n)}$ for each j). Write $\mathbf{a}_\infty^W = \mathbf{a}_\infty \cap S_\infty^W$, this is the ideal of S_∞^W generated by the elementary symmetric polynomials. There is also a natural W-action on A.
- We let $g = nq$ and $R_\infty = \mathcal{O}[\![x_1, \ldots, x_g]\!]$, and let $\mathfrak{q}_\infty = (x_1, \ldots, x_g) \in \operatorname{Spec}(R_\infty)$. For each N we have a map $R_\infty \to R_N$ such that $\mathfrak{q}_\infty R_N \subset \mathfrak{q}_N$ and $\mathfrak{q}_N/(\mathfrak{q}_N^2, \mathfrak{q}_\infty)$ is killed by a power of ϖ which is independent of N.
- Fix a non-principal ultrafilter $\mathcal{F}$ on $\mathbf{N}$, and let $\mathbf{R} = \prod_{N\in\mathbf{N}} \mathcal{O}$. If $I \in \mathcal{F}$, then we define $e_I = (\delta_{N\in I})_{N\in\mathbf{N}} \in \mathbf{R}$. Then $S = \{e_I \mid I \in \mathcal{F}\}$ is a multiplicative subset of $\mathbf{R}$, and we define $\mathbf{R}_\mathcal{F} = S^{-1}\mathbf{R}$. The natural map $\mathbf{R} \to \mathbf{R}_\mathcal{F}$ is surjective and factors through the projection $\prod_{N\geq 1} \mathcal{O} \to \prod_{N\geq m} \mathcal{O}$ for any $m \geq 1$. The ring $\mathbf{R}_\mathcal{F}$ can also be described as the localization of $\mathbf{R}$ at the prime ideal $\{(x_N)_{N\in\mathbf{N}} \mid \exists I \in \mathcal{F}, \forall N \in I, x_N \in \varpi\mathcal{O}\}$.

Definition 4.12. *We define*

- $M_1 = \varprojlim_m \left(\mathbf{R}_\mathcal{F} \otimes_\mathbf{R} \prod_{N\geq m} \left(S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q_N}}/\mathfrak{m}_{S_\infty}^m\right)\right)$
- $M_0 = \varprojlim_m \left(\mathbf{R}_\mathcal{F} \otimes_\mathbf{R} \prod_{N\geq m} S_\lambda(U_0(Q_N), \mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q_N}}\right)$
- $M = \varprojlim_m \left(\mathbf{R}_\mathcal{F} \otimes_\mathbf{R} \prod_{N\geq m} S_\lambda(U, \mathcal{O}/\varpi^m)_\mathfrak{m} \otimes_{A_{Q_N}^W} A_{Q_N}\right)$

Here $A_{Q_N}^W$ acts on $S_\lambda(U, \mathcal{O}/\varpi^m)_\mathfrak{m}$ via the spherical Hecke algebra action at places in Q_N. We note that we naturally obtain compatible actions of A on M, M_0 and M_1. Identifying $S_\lambda(U, \mathcal{O})_\mathfrak{m}$ with $\varprojlim_m \left(\mathbf{R}_\mathcal{F} \otimes_\mathbf{R} \prod_{N\geq m} S_\lambda(U, \mathcal{O}/\varpi^m)_\mathfrak{m}\right)$, we equip $S_\lambda(U, \mathcal{O})_\mathfrak{m}$ with an A^W action (A^W acts on the N factor in the product via its identification with $A_{Q_N}^W$) and we see that we have a natural isomorphism $M \cong S_\lambda(U, \mathcal{O})_\mathfrak{m} \otimes_{A^W} A$.

Lemma 4.13. *(1) M_1 is a flat S_∞-module.*
(2) The trace maps induce $M_1/\mathbf{a}_\infty \cong M_0$.
(3) We have a map $\eta : M \to M_0$ induced by the $\eta_{Q_N,m}$, which has kernel and cokernel killed by f, where $f = (f_{Q_N}^2) \in \prod_{N\in\mathbf{N}} R_N$ and f_{Q_N} is as in the statement of Proposition 4.8.

Proof. For the first two parts we apply [Pan19, Lemma 4.4.4(2)] (see also [Sta13, Tag 0912]): it suffices to prove that for each m

$$M_{1,m} = \mathbf{R}_\mathcal{F} \otimes_\mathbf{R} \prod_{N\geq m} \left(S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q_N}}/\mathfrak{m}_{S_\infty}^m\right)$$

is a flat $S_\infty/\mathfrak{m}_{S_\infty}^m$-module, the natural transition maps $M_{1,m+1} \to M_{1,m}$ induce isomorphisms $M_{1,m+1}/\mathfrak{m}_{S_\infty}^m \cong M_{1.m}$ and the trace maps induce $M_{1,m}/\mathbf{a}_\infty \cong M_{0,m}$.

Flatness of $M_{1,m}$ follows from flatness of $S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q_N}}$ over $\mathcal{O}[\Delta_N]$ (note that $S_\infty/\mathfrak{m}_{S_\infty}^N$ is a quotient of $\mathcal{O}[\Delta_N]$).

We have

$$M_{1,m+1}/\mathfrak{m}_{S_\infty}^m = \mathbf{R}_\mathcal{F} \otimes_\mathbf{R} \prod_{N\geq m+1} \left(S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q_N}}/\mathfrak{m}_{S_\infty}^m\right) = M_{1,m}$$

and

$$M_{1,m}/\mathbf{a}_\infty = \mathbf{R}_{\mathcal{F}} \otimes_{\mathbf{R}} \prod_{N\geq m} \Big(S_\lambda(U_1(Q_N),\mathcal{O})_{\mathfrak{m}_{1,Q_N}}/(\mathbf{a}_\infty + \mathfrak{m}_{S_\infty}^m)\Big)$$
$$= \mathbf{R}_{\mathcal{F}} \otimes_{\mathbf{R}} \prod_{N\geq m} \Big(S_\lambda(U_0(Q_N),\mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q_N}}\Big)$$

(see [Pan19, Lemma 4.5.9] for the first equalities).

The third part follows from [Pan19, Lemma 4.5.12]. □

Now we can define a patched pseudodeformation ring:

Definition 4.14. *For $m \geq 1$ we define $R^{\mathrm{p}}_m = \mathbf{R}_{\mathcal{F}} \otimes_{\mathbf{R}} \prod_{N\geq 1} R_N/(\mathfrak{m}_{R_N} f_{Q_N})^m$ and then define $R^{\mathrm{p}} = \varprojlim_m R^{\mathrm{p}}_m$.*

Lemma 4.15. *For each $m \geq 1$ there is an integer $n(m)$ (independent of N) such that $(\mathfrak{m}_{R_N} f_{Q_N})^{n(m)}$ annihilates $S_\lambda(U_1(Q_N),\mathcal{O})_{\mathfrak{m}_{1,Q_N}}/\mathfrak{m}^m_{S_\infty}$ for all $N \geq m$.*

Proof. By considering the $\mathbf{a}_\infty$-adic filtration on $S_\lambda(U_1(Q_N),\mathcal{O})_{\mathfrak{m}_{1,Q_N}}/\mathfrak{m}^m_{S_\infty}$ it suffices to prove that there is an integer $n(m)$ (independent of N) such that $(\mathfrak{m}_{R_N} f_{Q_N})^{n(m)}$ annihilates $S_\lambda(U_0(Q_N),\mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q_N}}$ for all $N \geq m$.

Since $f_{Q_N} S_\lambda(U_0(Q_N),\mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q_N}}$ is a finite length $\mathcal{O}$-module with length bounded by $qn!$ times that of $S_\lambda(U,\mathcal{O}/\varpi^m)_{\mathfrak{m}}$, its length as an R_N-module is bounded independently of N and therefore it is annihilated by a $\mathfrak{m}^{n(m)}_{R_N}$ for some $n(m)$ independent of N. □

It follows from Lemma 4.15 that R^{p} acts on M_1 (this is why R^{p} is defined the way it is). We are going to use [Pan19, Lemma 4.5.3] a few times, so we restate it here:

Lemma 4.16. *Suppose for $i \in \mathbf{N}$, M_i is an $\mathcal{O}$-module equipped with a decreasing filtration by $\mathcal{O}$-modules $M_i \supset M_{i,1} \supset M_{i,2} \cdots$. Then the natural map*

$$\prod_{i\geq 1} M_i \to \varprojlim_m \left(\mathbf{R}_{\mathcal{F}} \otimes_{\mathbf{R}} \prod_{i\geq 1} M_i/M_{i,m}\right)$$

is surjective with kernel given by elements of the form (m_i) such that for each m there exists $I_m \in \mathcal{F}$ with $m_i \in M_{i,m}$ for all $i \in I_m$.

We have a natural map $\prod_{N\geq 1} R_N \to R^{\mathrm{p}}$ which is surjective by Lemma 4.16. We also have a natural map $R^{\mathrm{p}} \to R_0$ given by taking the limit over m of

$$R^{\mathrm{p}}_m \to \mathbf{R}_{\mathcal{F}} \otimes_{\mathbf{R}} \prod_{N\geq 1} R_0/(\mathfrak{m}_{R_0})^m = R_0/(\mathfrak{m}_{R_0})^m.$$

Lemma 4.17. *The map $R^{\mathrm{p}} \to R_0$ we have just defined is surjective.*

Proof. We again apply Lemma 4.16: this implies that the natural map

$$\prod_{N\geq 1} R_N \to \varprojlim_m \left(\mathbf{R}_{\mathcal{F}} \otimes_{\mathbf{R}} \prod_{N\geq 1} R_0/(\mathfrak{m}_{R_0})^m\right)$$

is surjective; on the other hand it factors through our map $R^{\mathrm{p}} \to R_0$. □

From the n-dimensional pseudorepresentations $(\gamma_{N,j})_{N\geq 1}$ $(j = 1, \dots, q)$ with coefficients in $\prod_{N\geq 1} R_N$, we obtain n-dimensional pseudorepresentations $\gamma_{\infty,j}$ $(j = 1, \dots, q)$ of $\mathbf{Z}_p \times \mathbf{Z}$ with coefficients in R^{p}. On the other hand, M_1 has a natural structure of $S_\infty \otimes_{\mathcal{O}} A$-module, and we have defined characters $\alpha_j^{(i)} : \mathbf{Z}_p \times \mathbf{Z} \to (S_\infty \otimes_{\mathcal{O}} A)^\times$ and pseudocharacters $\alpha_j = \operatorname{tr} \alpha_j^{(1)} \oplus \dots \oplus \alpha_j^{(n)}$.

Lemma 4.18. *Fix $1 \leq i \leq q$.*

(1) *Composing $\gamma_{\infty,i}$ with the map $R^{\mathrm{p}} \to R_0$ gives a pseudorepresentation which is inflated from the 'unramified quotient' $\mathbf{Z}_p \times \mathbf{Z} \to \mathbf{Z}$ (i.e. projection to second factor).*

(2) *The composite of $\gamma_{\infty,j}$ with the map $R^{\mathrm{p}} \to \mathrm{End}(M_1)$ equals the composite of α_j with the map $S_\infty \otimes_{\mathcal{O}} A \to \mathrm{End}(M_1)$. Consequently, the map $R^{\mathrm{p}} \to \mathrm{End}(M_1)$ is a homomorphism of S_∞^W-algebras.*

Proof. The first part follows from the analogous statement for each of the pseudorepresentations $\gamma_{N,i}$ (which holds because the pseudorepresentations classified by R_0 are unramified at places in Q_N). The second part follows from Lemma 4.7. $\square$

Definition 4.19. *We let $\mathfrak{q}^{\mathrm{p}}$ be the prime ideal in R^{p} given by the inverse image of $\mathfrak{q}_0 \subset R_0$.*

Lemma 4.20. *The image of $\prod \mathfrak{q}_N \subset \prod_{N\geq 1} R_N$ in R^{p} is equal to $\mathfrak{q}^{\mathrm{p}}$.*

Proof. Write I for the kernel of $\prod_{N\geq 1} R_N \to R^{\mathrm{p}}$ and I' for the image of I in $\prod_{N\geq 1} R_N/\mathfrak{q}_N = \prod_{N\geq 1} \mathcal{O}$. It suffices to prove that the map $\prod_{N\geq 1} R_N \to R_0$ induces an isomorphism

$$\left(\prod_{N\geq 1} R_N\right)/(I, \prod \mathfrak{q}_N) = \left(\prod_{N\geq 1} \mathcal{O}\right)/I' \cong R_0/\mathfrak{q}_0 = \mathcal{O}.$$

The ideal I is the set of elements $(x_N) \in \prod R_N$ such that for each $m \geq 1$ there exists $I_m \in \mathcal{F}$ with $x_N \in (\mathfrak{m}_{R_N} f_{Q_N})^m$ for all $N \in I_m$. We have $(\mathfrak{m}_{R_N} f_{Q_N})^m + \mathfrak{q}_N = (\varpi^m f_{Q_N}^m) + \mathfrak{q}_N \subset (\varpi^m) + \mathfrak{q}_N$. It follows that I' is contained in the kernel of the map $\prod_{N\geq 1} \mathcal{O} \to \mathcal{O}$. We need to show that I' equals the kernel. To this end, choose a tuple of elements $(y_N) \in \prod_{N\geq 1} \mathcal{O}$ which does lie in the kernel. Recall that there is a constant c such that the image of f_{Q_N} in $R_N/\mathfrak{q}_N = \mathcal{O}$ has ϖ-adic valuation $\leq c$. Let $I_m = \{N \geq 1 \mid \mathrm{ord}_\varpi y_N \geq m(c+1)\}$. Then $I_1 \supset I_2 \supset I_2 \supset \dots$ and $\cap_{m\geq 1} I_m = \emptyset$. Moreover, each I_m is in $\mathcal{F}$ (since (y_N) is in the kernel of the map to $\mathcal{O}$).

We have $(\mathfrak{m}_{R_N} f_{Q_N})^m + \mathfrak{q}_N = (\varpi^m f_{Q_N}^m) + \mathfrak{q}_N$, and this contains $(\varpi^{(c+1)m}) + \mathfrak{q}_N$. Therefore we can for each $m \geq 1$ and $N \in I_m$ find an element $x_{N,m} \in (\mathfrak{m}_{R_N} f_{Q_N})^m$ such that $x_{N,m} + \mathfrak{q}_N = y_N$. We define a tuple $(x_N)_{N\geq 1} \in \prod_{N\geq 1} R_N$ by taking x_N to be an arbitrary pre-image of y_N if $N \notin I_1$ and $x_N = x_{N,m}$ if $N \in I_m - I_{m+1}$. Then (x_N) lies in I and its image in $\prod_{N\geq 1} \mathcal{O}$ equals (y_N), as required. $\square$

Lemma 4.21.

(1) *We have an equality of ideals $\prod_{N\geq 1} \mathfrak{q}_N^m = \left(\prod_{N\geq 1} \mathfrak{q}_N\right)^m$ in $\prod_{N\geq 1} R_N$.*

(2) *For $m \geq 1$ the image of $\prod_{N\geq 1} \mathfrak{q}_N^m$ in R^{p} is equal to $(\mathfrak{q}^{\mathrm{p}})^m$.*

The possibility of proving a statement like this one is mentioned in [Pan19, Remark 4.6.10].

Proof. It suffices to prove the first part. Recall from the proof of Corollary 2.31 that there exists an integer g_0 such that for every N there exists a surjection $\mathcal{O}[\![x_1,\dots,x_{g_0}]\!] \to R_N$ such that the images of the x_i are in $\mathfrak{q}_N$ (since $\mathfrak{q}_N/\mathfrak{q}_N^2$ can be generated by g_0 elements). Now it suffices to prove that we have an equality of ideals $\prod_{N\geq 1}(x_1,\dots,x_{g_0})^m = \left(\prod_{N\geq 1}(x_1,\dots,x_{g_0})\right)^m$ in $\prod_{N\geq 1}\mathcal{O}[\![x_1,\dots,x_{g_0}]\!]$. We conclude using the fact that for any ring R and any ideal $I \subset R$ we have $I^m\prod_{N\geq 1}R = \left(I\prod_{N\geq 1}R\right)^m$, and we also have $I\prod_{N\geq 1}R = \prod_{N\geq 1}I$ when I is finitely generated. □

For the statement of the next proposition, we recall that our data includes, for each $N \geq 1$, a map $R_\infty = \mathcal{O}[\![x_1,\dots,x_g]\!] \to R_N$ sending the ideal $\mathfrak{q}_\infty = (x_1,\dots,x_g)$ to $\mathfrak{q}_N$. The diagonal map $R_\infty \to \prod_{N\geq 1}R_N$ induces a map $R_\infty \to R^{\mathrm{p}}$ which sends $\mathfrak{q}_\infty$ into $\mathfrak{q}^{\mathrm{p}}$.

Proposition 4.22. *(1) The $\mathcal{O}$-module*

$$\mathfrak{q}^{\mathrm{p}}/((\mathfrak{q}^{\mathrm{p}})^2,\mathfrak{q}_\infty)$$

is killed by ϖ^c, where c is as in Corollary 2.31.

(2) The natural map on completed local rings

$$(R_\infty)\widehat{\ }_{\mathfrak{q}_\infty} \to (R^{\mathrm{p}})\widehat{\ }_{\mathfrak{q}^{\mathrm{p}}}$$

is surjective. In particular, $(R^{\mathrm{p}})\widehat{\ }_{\mathfrak{q}^{\mathrm{p}}}$ is a complete Noetherian local E-algebra with residue field E.

Proof. It follows from Corollary 2.31 that the cokernel of the map

$$\prod_{N\geq 1}\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \prod_{N\geq 1}\mathfrak{q}_N/(\mathfrak{q}_N)^2$$

is killed by ϖ^c. Applying Lemmas 4.20 and 4.21, it remains to show that the image of $\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2$ in $\mathfrak{q}^{\mathrm{p}}/(\mathfrak{q}^{\mathrm{p}})^2$ is the same as the image of $\prod_{N\geq 1}\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2$. This is done as in the proof of [Pan19, Prop. 4.6.16]: it suffices to show that the composition of maps

$$\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \prod_{N\geq 1}\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \mathcal{O}\otimes_{\prod_{N\geq 1}\mathcal{O}}\prod_{N\geq 1}\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2$$

is surjective. Here the first map is the diagonal embedding and we regard $\mathcal{O}$ as a $\prod_N\mathcal{O}$-algebra via the map $\prod_N\mathcal{O} \to R^{\mathrm{p}}/\mathfrak{q}^{\mathrm{p}} \cong \mathcal{O}$. We conclude using Lemma 4.23.

This shows the first part of the proposition. For the second, we see that the first part implies that each of the maps

$$g_i : (R_\infty/\mathfrak{q}_\infty^i)_{\mathfrak{q}_\infty} \to (R^{\mathrm{p}}/(\mathfrak{q}^{\mathrm{p}})^i)_{\mathfrak{q}^{\mathrm{p}}}$$

is surjective. To check that $\varprojlim_i g_i$ is surjective, it is enough to note that the sequence $(\ker g_i)_{i\geq 1}$ satisfies the Mittag-Leffler condition (because each of these ideals has finite length, being contained in an Artinian local ring). □

Lemma 4.23. *Let R be a commutative ring and M a finitely generated R-module. Suppose we have a R-algebra map $\prod_{N\geq 1}R \xrightarrow{\lambda} R$. Then the composite map*

$$M \to \prod_{N\geq 1}M \to R\otimes_{\prod_{N\geq 1}R}\prod_{N\geq 1}M$$

is surjective.

Proof. If M is finite free over R, then $\prod_{N\geq 1} M$ has a $(\prod_N R)$-basis given by diagonally embedded basis elements for M, and the statement is clear. In general, we write M as a quotient of a finite free R-module F. The composition of $F \to R\otimes_{\prod_{N\geq 1} R}\prod_{N\geq 1} F \to R\otimes_{\prod_{N\geq 1} R}\prod_{N\geq 1} M$ is surjective and factors through M. □

Remark 4.24. Note that we have not shown that $\mathfrak{q}^{\mathrm{p}}$ is finitely generated, so we rely on the comparison with R_∞ to show that $\mathfrak{q}^{\mathrm{p}}$-adic completion $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$ is $\mathfrak{q}^{\mathrm{p}}$-adically complete!

Now we are going to consider the modules:

- $\mathrm{m}_1 = (M_1/\mathbf{a}_\infty^2)_{\mathfrak{q}^{\mathrm{p}}}$
- $\mathrm{m}_0 = (M_0)_{\mathfrak{q}^{\mathrm{p}}}$
- $\mathrm{m} = M_{\mathfrak{q}^{\mathrm{p}}} = M_{\mathfrak{q}_0}$.

Lemma 4.25. *(1)* m_1 *is a finite free* $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$*-module.*
(2) The trace maps induce an isomorphism $\mathrm{m}_1/\mathbf{a}_\infty \cong \mathrm{m}_0$.
(3) The map η *induces an isomorphism* $\eta : \mathrm{m} \cong \mathrm{m}_0$.

Proof. We start with the third part: this follows immediately from Lemma 4.13, since by Proposition 4.8 the image of f in R^{p} is not in $\mathfrak{q}^{\mathrm{p}}$. The second part also follows immediately from Lemma 4.13. It remains to show the first part. Since the inverse image of $\mathfrak{q}^{\mathrm{p}}$ in S_∞^W is $\mathbf{a}_\infty^W$, the action of S_∞ on m_1 factors through the localisation $S_\infty \otimes_{S_\infty^W} (S_\infty^W)_{\mathbf{a}_\infty^W} = S_{\infty,\mathbf{a}_\infty}$ (note that $\mathbf{a}_\infty$ is the unique point of $\mathrm{Spec}(S_\infty)$ in the pre-image of $\mathbf{a}_\infty^W$ under the finite map $\mathrm{Spec}(S_\infty) \to \mathrm{Spec}(S_\infty^W)$). We know from Lemma 4.13 that $M_1/\mathbf{a}_\infty^2$ is a flat $S_\infty/\mathbf{a}_\infty^2$-module, so the localisation m_1 is a flat $S_\infty/\mathbf{a}_\infty^2$-module, and hence a flat $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$-module. Since $\mathrm{m}_1/\mathbf{a}_\infty$ is finite dimensional (combining the second and third parts), m_1 is finitely generated over the Artinian local ring $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$. □

Since m_1 is a finite dimensional E-vector space, the action of the local E-algebra $(R^{\mathrm{p}})_{\mathfrak{q}^{\mathrm{p}}}$ factors through an action by (an Artinian quotient of) $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$. It follows from the third part of Lemma 4.25 that the action of $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$ on $\mathrm{m}_0 \cong \mathrm{m}$ factors through the composition of surjective maps

$$(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}} \to (R_0)^\wedge_{\mathfrak{q}_0} \to (\mathbf{T}_\emptyset)_{\mathfrak{q}_0} = E \tag{4.25.1}$$

Now we consider again our pseudorepresentations $\gamma_{\infty,j}$ $(1 \leq j \leq q)$ of $\mathbf{Z}_p \times \mathbf{Z}$ with coefficients in R^{p}.

Definition 4.26. *For* $1 \leq j \leq q$*, we let* $\delta_j \in R^{\mathrm{p}}$ *denote the discriminant of the characteristic polynomial* $\chi_j(t) \in R^{\mathrm{p}}[t]$ *of* $(0,1) \in \mathbf{Z}_p \times \mathbf{Z}$ *under the pseudorepresentation* $\gamma_{\infty,j}$.

Lemma 4.27. *For* $1 \leq j \leq q$*,* $\delta_j \notin \mathfrak{q}^{\mathrm{p}}$*. Moreover,* $\chi_j(t)$ *mod* $\mathfrak{q}^{\mathrm{p}}$ *splits into linear factors in* $E[t]$.

Proof. To show that $\delta_j \neq 0$, it suffices to show that for some $m \geq 1$ the image of δ_j under the composition

$$R^{\mathrm{p}} \to R_0 \overset{h_{U,\sigma}}{\to} \mathcal{O} \to \mathcal{O}/\varpi^m$$

is non-zero. Recall the constant d from Lemma 2.30. Choose $m > dn(n-1)$. Then it follows from Lemma 2.30 that we will be done if we can identify the image of δ_j in $\mathcal{O}/\varpi^m$ with the image of the discriminant of the characteristic polynomial of a Frobenius element $\sigma_{\tilde{v}}$ for some $v \in Q_N$. Choose m' so that our map $R_0 \to \mathcal{O}/\varpi^m$

factors through $R_0/\mathfrak{m}_{R_0}^{m'}$. Now we can identify the image of δ_j in $R_0/\mathfrak{m}_{R_0}^{m'}$ with the image of an element $(\delta_{j,N})_{N\geq 1} \in \prod_{N\geq 1} R_0/\mathfrak{m}_{R_0}^{m'}$ in $\mathbf{R}_{\mathcal{F}} \otimes_{\mathbf{R}} \prod_{N\geq 1} R_0/\mathfrak{m}_{R_0}^{m'}$, where $\delta_{j,N}$ is the image of the discriminant for the Frobenius element at the jth element of Q_N. We deduce that the image of δ_j in $R_0/\mathfrak{m}_{R_0}^{m'}$ coincides with one of these Frobenius discriminants.

The same argument shows that the image of $\chi_j(t) \bmod \mathfrak{q}^{\mathrm{p}}$ splits into linear factors in $\mathcal{O}/\varpi^m[t]$ for all $m \geq 1$. Indeed, for each $\sigma_{\widetilde{v}} \in G_F$, the characteristic polynomial of $\rho(\sigma_{\widetilde{v}})$ has all of its roots in $\mathcal{O}$ (this is part of the definition of an enormous subgroup of $\mathrm{GL}_n(\mathcal{O})$). Hensel's lemma implies that $\chi_j(t)$ itself factors in $\mathcal{O}[t]$. $\square$

For each $j \in \{1, \ldots, q\}$ we fix an ordering $x_j^{(1)}, \ldots, x_j^{(n)}$ of the (pairwise distinct) roots in E of the polynomial $\chi_j(t) \bmod \mathfrak{q}^{\mathrm{p}}$. For each j, we may consider the pseudorepresentation $(\gamma_{\infty,j})_{\mathfrak{q}^{\mathrm{p}}}$ of $\mathbf{Z}_p \times \mathbf{Z}$ with coefficients in $(R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}}$ given by composing $\gamma_{\infty,j}$ with the natural map $R^{\mathrm{p}} \to (R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}}$. This pseudorepresentation is residually multiplicity free.

Lemma 4.28. *There is a unique collection of continuous characters $\gamma_j^{(i)} : \mathbf{Z}_p \times \mathbf{Z} \to ((R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}})^{\times}$ ($i = 1, \ldots, n$, $j = 1, \ldots, q$) such that $\gamma_j^{(i)} \bmod \mathfrak{q}^{\mathrm{p}}$ is the character $(a, b) \to (x_j^{(i)})^b$ and $(\gamma_{\infty,j})_{\mathfrak{q}^{\mathrm{p}}} = \operatorname{tr} \gamma_j^{(1)} \oplus \cdots \oplus \gamma_j^{(n)}$.*

Proof. This follows from, e.g., [BC09, Proposition 1.5.1], since in a commutative GMA we have (using the notation of *loc.cit.*) $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = \mathcal{A}_{j,i}\mathcal{A}_{i,j} \subset \mathcal{A}_{i,i} \cap \mathcal{A}_{j,j} = 0$ for $i \neq j$. $\square$

The characters $\gamma_j^{(i)}|_{\mathbf{Z}_p \times 0} : \mathbf{Z}_p \to ((R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}})^{\times}$ determine an extension of the homomorphism $S_\infty^W \to R^{\mathrm{p}}$ to a homomorphism $S_\infty \to (R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}}$. This in turn naturally extends to a map from the formally smooth E-algebra $(S_\infty)^{\wedge}_{\mathbf{a}_\infty}$ and we choose a lift of this through the surjective map (see Proposition 4.22)

$$(R_\infty)^{\wedge}_{\mathfrak{q}_\infty} \to (R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}}$$

to equip $(R_\infty)^{\wedge}_{\mathfrak{q}_\infty}$ with a map from $(S_\infty)^{\wedge}_{\mathbf{a}_\infty}$. We denote by A' the localization of A at the prime ideal $(t_j^{(i)} - x_j^{(i)} : 1 \leq j \leq q, 1 \leq i \leq n)$ and define

- $\mathrm{m}_1' = \mathrm{m}_1 \otimes_A A'$
- $\mathrm{m}_0' = \mathrm{m}_0 \otimes_A A'$.

(We recall that the ring A, defined at the beginning of §4.11, is a Laurent polynomial ring in elements $(t_j^{(i)})^{\pm 1}$ ($j = 1, \ldots, q$, $i = 1, \ldots, n$) which represent the patched version of the Hecke operators $t_{v,i}(\varpi_v)^{\pm 1}$ for Taylor–Wiles primes v.)

Remark 4.29. The above localization is our replacement for the usual 'localization with respect to a suitable eigenvalue of the U_q operator' which appears in the Taylor–Wiles method. We can only do this after patching and inverting p because we do not assume that $\overline{\rho}(\sigma_{\widetilde{v}})$ has distinct eigenvalues for Taylor–Wiles places v.

Lemma 4.30. *(1) For each $i = 1, \ldots, n$ and $j = 1, \ldots, q$, the respective push-forwards of the characters $\alpha_j^{(i)}$, $\gamma_j^{(i)}$ to $\mathrm{End}(\mathrm{m}_1')$ are equal.*

(2) The two structures of S_∞-module on m_1' (the standard one, and the one induced by the homomorphism $S_\infty \to (R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}}$ constructed above) are the same.

(3) The map $(R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}} \to (R_0)^{\wedge}_{\mathfrak{q}_0}$ factors through the quotient $(R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}^{\mathrm{p}}}/\mathbf{a}_\infty$.

(4) The trace maps induce $m_1'/\mathbf{a}_\infty \cong m_0'$.
(5) m_1' *is a finite free (non-zero)* $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$*-module.*

Proof. Let $X = \{\alpha_j^{(i)}(z), \gamma_j^{(i)}(z) \mid z \in \mathbf{Z}_p \times \mathbf{Z}, i = 1, \dots, n,\ j = 1, \dots, q\}$. By construction, the elements of X commute with each other; let T denote the E-subalgebra of $\mathrm{End}(m_1')$ generated by the elements of X. Then T is an Artinian E-algebra. The pushforwards of the characters $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$ take values in T and the pseudocharacters $\mathrm{tr}\,\alpha_j^{(1)} \oplus \cdots \oplus \alpha_j^{(n)}$ and $\mathrm{tr}\,\gamma_j^{(i)} \oplus \cdots \oplus \gamma_j^{(n)}$ are equal after pushforward to T for each $j = 1, \dots, q$. To show that the characters $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$ are equal after pushforward to T for each $j = 1, \dots, q$ it is enough (after the uniqueness assertion of [BC09, Proposition 1.5.1]) to show that they are equal after pushforward to each residue field of T. However, our construction shows that for each $i = 1, \dots, n$ and $j = 1, \dots, q$ the elements $\alpha_j^{(i)}(0,1) - x_j^{(i)}$ and $\gamma_j^{(i)}(0,1) - x_j^{(i)}$ are commuting nilpotent elements of $\mathrm{End}(m_1')$ and therefore their difference $\alpha_j^{(i)}(0,1) - \gamma_j^{(i)}(0,1)$ lies in the Jacobson radical of T. This proves the first part of the lemma. The second is an immediate consequence since the two S_∞-module structures are determined by the two sets of characters $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$.

The third part of the lemma is equivalent to the assertion that characters $\gamma_j^{(i)}|_{\mathbf{Z}_p \times 0}$ become trivial after pushforward along the map $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}} \to (R_0)^\wedge_{\mathfrak{q}_0}$. Since the pseudocharacter $\mathrm{tr}\,\gamma_j^{(1)} \oplus \cdots \oplus \gamma_j^{(n)}$ is residually multiplicity-free, the desired statement follows from uniqueness and Lemma 4.18.

The fourth part of the lemma follows from the same statement before localisation to A' (Lemma 4.25). We now prove the final part of the lemma. Since m_1' is a direct summand of m_1, we just need to prove that m_1' is non-zero, or indeed that m_0' is non-zero. For this, we note that it follows from Lemma 4.18 (compatibility of Galois and automorphic pseudocharacters) and the observation above (4.25.1) that the characteristic polynomial $\prod_{i=1}^n (t - t_j^{(i)})$ of $(0,1)$ under α_j pushes forwards to $\prod_{i=1}^n (t - x_j^{(i)}) = \chi_j(t) \bmod \mathfrak{q}^{\mathrm{p}}$ in $\mathrm{End}(m_0)$. It follows that A^W acts on m_0 via the map $A^W \to E$ induced by $t_j^{(i)} \mapsto x_j^{(i)}$. Since the A-module m_0 is isomorphic to $S_\lambda(U, \mathcal{O})_{\mathfrak{q}_0} \otimes_{A^W} A$, we deduce that the localisation m_0' is non-zero. $\square$

To complete the proof of this section's main theorem, we recall Brochard's freeness criterion:

Theorem 4.31 (Theorem 1.1 of [Bro17])**.** *Let $A \to B$ be a local morphism of Noetherian local rings satisfying the inequality on embedding dimensions:*

$$\mathrm{edim}(B) \leq \mathrm{edim}(A).$$

Let M be a non-zero A-flat B-module which is finitely generated over B. Then M is finite free over B.

Theorem 4.32. *The map $(R_0)^\wedge_{\mathfrak{q}^0} \to (\mathbf{T}_\emptyset)_{\mathfrak{q}_0} = E$ is an isomorphism, and as a consequence we have*

$$H^1_f(F^+, \mathrm{ad}\, r_{\pi,\iota}) = 0.$$

Proof. We apply Brochard's criterion with $A = S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$, $B = (R_\infty)^\wedge_{\mathfrak{q}_\infty}/\mathbf{a}_\infty^2$, $M = m_1'$. Note that the embedding dimension of $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$ is qn and, since $(R_\infty)^\wedge_{\mathfrak{q}_\infty}$ is a power series ring over E in qn variables, the embedding dimension of $(R_\infty)^\wedge_{\mathfrak{q}_\infty}/\mathbf{a}_\infty^2$ is $\leq qn$.

We conclude that m_1' is finite free over $(R_\infty)^\wedge_{\mathfrak{q}_\infty}/\mathbf{a}_\infty^2$ and therefore m_0' is finite free over $(R_\infty)^\wedge_{\mathfrak{q}_\infty}/\mathbf{a}_\infty$. Since the action of $(R_\infty)^\wedge_{\mathfrak{q}_\infty}$ on m_0' factors through the action of $(\mathbf{T}_\emptyset)_{\mathfrak{q}_0}$, we deduce that each of the surjective maps

$$(R_\infty)^\wedge_{\mathfrak{q}_\infty}/\mathbf{a}_\infty \to (R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}/\mathbf{a}_\infty \to (R_0)^\wedge_{\mathfrak{q}_0} \to (\mathbf{T}_\emptyset)_{\mathfrak{q}_0} = E$$

are isomorphisms. The vanishing of the adjoint Selmer group follows from the identification of this with the reduced tangent space of $(R_0)^\wedge_{\mathfrak{q}_0}$ (i.e. Proposition 2.21). ☐

Remark 4.33. We find it convenient (or amusing) to use Brochard's freeness criterion here, although we could alternatively have worked with the $(S_\infty)^\wedge_{\mathbf{a}_\infty}$-module $\varprojlim_m ((M_1/\mathbf{a}_\infty^m)_{\mathfrak{q}^{\mathrm{p}}})$ in place of m_1 and concluded using Auslander–Buchsbaum as in the work of Diamond and Fujiwara.

5. Applications

We now deduce our main theorems. We begin with a useful lemma.

Lemma 5.1. *Let F be a number field, and let $E/\mathbf{Q}_p$ be a coefficient field. Let $\rho : G_F \to \mathrm{GL}_n(E)$ be a continuous representation which is unramified almost everywhere. Let S be a finite set of places of F. Then we can find a finite set T of places of F with the following property:*

- *$T \cap S = \emptyset$.*
- *For any T-split finite extension F'/F, $\rho(G_{F'(\zeta_{p^\infty})}) = \rho(G_{F(\zeta_{p^\infty})})$.*

Proof. After replacing ρ by $\rho \oplus \epsilon$, it is enough to show we can choose T so that $\rho(G_F) = \rho(G_{F'})$ if F'/F is T-split. Conjugate ρ so that it takes values in $\mathrm{GL}_n(\mathcal{O})$, and let L_∞/F be the extension cut out by ρ, L_N/F the extension cut out by $\rho_N = \rho \bmod \varpi^N$. We have $\rho(G_F) = \varprojlim_N \rho_N(G_F)$, so it's enough to show that we can choose T so that if F'/F is T-split, then $\rho_N(G_F) = \rho_N(G_{F'})$ for all $N \geq 1$.

To this end, we let M/F be the compositum of all of the extensions of F cut out by simple quotients of $\mathrm{Gal}(L_N/F)$ (for any $N \geq 1$). The extension M/F is finite, because simple quotients of $\mathrm{Gal}(L_N/F)$ (for varying $N \geq 1$) correspond to simple quotients of $\rho(G_F)$ by closed normal subgroups. Since $\rho(G_F)$ has a normal, closed subgroup of finite index which is a topologically finitely generated pro-p group, these quotients are finite in number.

We can therefore choose T to be any set disjoint from S and such that for each intermediate field $M/M'/F$ with $\mathrm{Gal}(M'/F)$ simple, there exists $v \in T$ which does not split in M'. ☐

We prove a theorem for regular algebraic, cuspidal, polarized automorphic representations. First we treat the case of a CM base field.

Theorem 5.2. *Let F be a CM number field, and let (π, χ) be a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, and suppose that $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous. Then $H^1_f(F^+, \operatorname{ad} r_{\pi,\iota}) = 0$.*

Proof. As in the proof of [BLGHT11, Theorem 1.2], π has a twist which is polarized with respect to δ^n_{F/F^+} (i.e. of unitary type). The twist does not alter $\operatorname{ad} r_{\pi,\iota}$, so we can assume that π is of unitary type. For any finite extension F'/F^+, the induced map

$$H^1_f(F^+, \operatorname{ad} r_{\pi,\iota}) \to H^1_f(F', \operatorname{ad} r_{\pi,\iota})$$

is injective. It is therefore enough to find a soluble totally real extension L^+/F^+ with the following properties:

- Let $L = L^+F$. Then $r_{\pi,\iota}(G_{L(\zeta_{p^\infty})}) = r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$.
- Let π_L denote the base change of π (which exists and is regular algebraic, after [AC89, Ch. 3, Theorems 4.2, 5.1]). It is cuspidal, because $r_{\pi,\iota}|_{G_L}$ is irreducible. Each place of L at which π_L is ramified, or dividing p, is split over L^+.
- For every place w of L, $\pi_{L,w}$ has an Iwahori–fixed vector.

To achieve this, let S be the set of places of F^+ dividing p or above which π is ramified, and let S_F denote the set of places of F lying above a place of S. Let T_F denote a set as provided by Lemma 5.1, disjoint from S_F, and let T denote the set of places of F^+ lying below a place of T_F. Then S and T are disjoint and if L^+/F^+ is T-split, then L/F is T_F-split. We can choose L^+/F^+ to be any T-split soluble totally real extension which has the correct behaviour at the places in S, the existence of such an extension being a consequence of [CHT08, Lemma 4.1.2]. □

Next we treat the case of a totally real base field F. We consider a regular algebraic, cuspidal, polarized automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbf{A}_F)$. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, and suppose that $r_{\pi,\iota}$ is irreducible. Let V denote the space on which $r_{\pi,\iota}$ acts. Then there is a unique G_F-equivariant pairing $\langle\cdot,\cdot\rangle : V \times V \to \epsilon^{1-n} r_{\chi,\iota}$, which is symmetric if n is odd or n is even and $r_{\chi,\iota}(c_v) = 1$ ($v|\infty$), or antisymmetric if n is even and $r_{\chi,\iota}(c_v) = -1$ (see [BC11] and [BLGGT14, §2.1]). We thus obtain a homomorphism

$$r'_{\pi,\iota} : G_F \to \mathrm{GS}(\langle\cdot,\cdot\rangle)(\overline{\mathbf{Q}}_p)$$

to the general similitude group of the pairing $\langle\cdot,\cdot\rangle$. We write $\mathfrak{gs}$ for the Lie algebra of this reductive group over $\overline{\mathbf{Q}}_p$.

Theorem 5.3. *Let F be a totally real number field, and let (π, χ) be a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, and suppose that $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous. Then $H^1_f(F, \mathfrak{gs}) = 0$.*

Proof. This can be deduced from Theorem 5.2 using base change in the same way that [All16, Theorem B] is deduced from [All16, Theorem A]. We omit the details. □

When $n = 2$, these results take a particularly simple form:

Theorem 5.4. *Let F be a totally real number field, and let π be a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Suppose that one of the following holds:*

(1) π does not have CM.

(2) π has CM by an extension K/F, and $K \not\subset F(\zeta_{p^\infty})$.

Then $H^1_f(F, \operatorname{ad} r_{\pi,\iota}) = 0$.

Proof. When $n = 2$, $\mathfrak{gs} = \mathfrak{gl}_2$. Our result will follow from Theorem 5.3 if we can verify that our hypotheses imply that $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous. If π does not have CM, this is example 2.34.

Suppose instead that π has CM by a CM quadratic extension K/F, and K is not contained in $F(\zeta_{p^\infty})$. To show that $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous, it is enough to show that we can find regular semisimple elements in the image of both $G_{K(\zeta_{p^\infty})}$ and $G_{F(\zeta_{p^\infty})} - G_{K(\zeta_{p^\infty})}$. Elements of the latter type exist because of our assumption that K is not contained in $F(\zeta_{p^\infty})$.

Now suppose for a contradiction that $r_{\pi,\iota} \cong \operatorname{Ind}_{G_K}^{G_F} \chi$ is scalar on restriction to $G_{K(\zeta_{p^\infty})}$. This implies that χ/χ^c is trivial on $G_{K(\zeta_{p^\infty})}$, and hence that $(\chi/\chi^c)^2 = 1$ (since c acts trivially on $\operatorname{Gal}(K(\zeta_{p^\infty})/K)$). This contradicts the fact that χ/χ^c has infinite order (because its Hodge–Tate weights are all non-zero, because π is regular algebraic). This completes the proof. $\square$

We can also prove results for elliptic curves.

Theorem 5.5. *Let F be a totally real number field, and let E be an elliptic curve over F. Let p be a prime, and suppose that one of the following holds:*

(1) E does not have CM.

(2) E has CM by a quadratic field $K/\mathbf{Q}$, and $K \not\subset F(\zeta_{p^\infty})$.

Then $H^1_f(F, \operatorname{ad} V_p(E)) = 0$.

Proof. If the elliptic curve E has CM, then its p-adic Galois representations are automorphic and we can appeal to Theorem 5.4. If E does not have CM, then there exists a totally real extension F'/F such that the p-adic Galois representations of $E_{F'}$ are automorphic (for example, by [Tay06]) and we can appeal again to the same theorem. $\square$

Combining our results with potential automorphy theorems, we can prove some more general vanishing results. Here is an example for symmetric powers of two-dimensional represention.

Theorem 5.6. *Let F be a CM number field, and let (π, χ) be a regular algebraic, cuspidal, polarized automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ such that $\operatorname{Sym}^2 \pi$ is cuspidal. Let p be a prime, and fix an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. Then for any $n \geq 1$, $H^1_f(F^+, \operatorname{ad} \operatorname{Sym}^{n-1} r_{\pi,\iota}) = 0$.*

Proof. By [BLGGT14, Theorem 5.4.1], there exists a Galois, CM extension F'/F such that $\operatorname{Sym}^{n-1} r_{\pi,\iota}|_{G_{F'}}$ is automorphic. It suffices to show the vanishing of $H^1_f((F')^+, \operatorname{ad} \operatorname{Sym}^{n-1} r_{\pi,\iota})$, and this follows from Theorem 5.2 once we verify that $\operatorname{Sym}^{n-1} r_{\pi,\iota}(G_{F'(\zeta_{p^\infty})})$ is enormous. However, this follows from Example 2.34. $\square$

Finally, we give an application to vanishing results for anticyclotomic characters, as predicted by the Bloch–Kato conjecture. Over a general CM field our main theorem gives vanishing results which are not covered by known cases of the anticyclotomic main conjecture (cf. those proved in [Hid09]).

Theorem 5.7. *Let F be a CM number field, and let $\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ be a unitary character of type A_0. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, and suppose that the following conditions are satisfied:*

(1) $\chi\chi^c = 1$.

(2) The integers n_τ ($\tau \in \mathrm{Hom}(F, \mathbf{C})$) defined by $\chi_v(z) = \tau(z)^{n_\tau} \tau c(z)^{n_{\tau c}}$ for each place $v|\infty$ are of constant parity, and none of them are zero.

(3) $F \not\subset F^+(\zeta_{p^\infty})$.

Then $H^1_f(F, r_{\chi,\iota}) = 0$.

Proof. The given conditions imply that there is a character $\psi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ of type A_0 such that $\psi/\psi^c = \chi$. Given this, let π denote the automorphic induction of ψ from F to F^+. It is a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{F^+})$ and $\mathrm{Ind}_{G_F}^{G_{F^+}} r_{\chi,\iota}$ is a subquotient of $\mathrm{ad}\, r_{\pi,\iota}$, so the desired vanishing follows from Shapiro's lemma for Bloch–Kato Selmer groups and Theorem 5.4.

Let us explain why ψ exists. Choose arbitrarily integers m_τ such that $2m_\tau + n_\tau = w$, is independent of τ. Then $m_{\tau c} = w - m_\tau$, so there exists a character $\mu : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ of type A_0 such that $\mu_v(z) = \tau(z)^{m_\tau} \tau c(z)^{m_{\tau c}}$. Moreover $m_\tau - m_{\tau c} = -n_\tau$, so $\chi_0 = \chi(\mu/\mu^c)$ has finite order and satisfies $\chi_0 \chi_0^c = 1$. Lemma 5.8 implies that there exists another finite order Hecke character ϕ such that $\phi/\phi^c = \chi_0$, so we can then take $\psi = \phi\mu^{-1}$. □

Lemma 5.8. *Let F be a CM field, and let $\chi : G_F \to \mathbf{Q}/\mathbf{Z}$ be a continuous character such that $\chi\chi^c = 1$. Then there exists a continuous character $\phi : G_F \to \mathbf{Q}/\mathbf{Z}$ such that $\phi/\phi^c = \chi$.*

Proof. It is equivalent to ask that $H^1(F/F^+, H^1(F, \mathbf{Q}/\mathbf{Z})) = 0$. We use the Hochschild–Serre spectral sequence

$$H^p(F/F^+, H^q(F, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(F^+, \mathbf{Q}/\mathbf{Z}).$$

We recall that if K is a number field, then the product

$$H^r(K, \mathbf{Q}/\mathbf{Z}) \to \prod_{v|\infty} H^r(K_v, \mathbf{Q}/\mathbf{Z})$$

of restriction maps is an isomorphism when $r \geq 3$ ([Mil06, Theorem 4.20]) and that $H^2(K, \mathbf{Q}/\mathbf{Z}) = 0$ (Tate's theorem, see [Ser77, Theorem 4] or [Pat19, Theorem 2.1.1]). Since F has no real places, the groups $H^r(F, \mathbf{Q}/\mathbf{Z})$ vanish when $r \geq 2$ and so the spectral sequence in question has only two rows, and can be pieced together into a long exact sequence (cf. [Wei94, Exercise 5.2.2]) including the terms

$$\begin{tikzcd} H^2(F^+, \mathbf{Q}/\mathbf{Z}) \arrow[r] & H^1(F/F^+, H^1(F, \mathbf{Q}/\mathbf{Z})) \arrow[d] & \\ & H^3(F/F^+, H^0(F, \mathbf{Q}/\mathbf{Z})) \arrow[r] & H^3(F^+, \mathbf{Q}/\mathbf{Z}). \end{tikzcd}$$

The edge morphism $H^3(F/F^+, H^0(F, \mathbf{Q}/\mathbf{Z})) \to H^3(F^+, \mathbf{Q}/\mathbf{Z})$ is inflation, and is injective because the extension F/F^+ is CM and the map $H^3(F^+, \mathbf{Q}/\mathbf{Z}) \to \prod_{v|\infty} H^3(F_v^+, \mathbf{Q}/\mathbf{Z})$ is bijective. This completes the proof. □

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